Valuing barrier options using the adaptive discontinuous Galerkin method

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Outline

Introduction

- **Financial problem**:

  How to calculate a fair value of the option?

- **Classical market model**: Black-Scholes equation
- **Mathematical point of view**:
  scalar nonstationary convection–diffusion–reaction equation with linear convection and linear diffusion,
- **Numerical method**:

  - discontinuous Galerkin method (DGM) with IPG variants: nonsymmetric, symmetric and incomplete,
  - piecewise polynomial discontinuous approximation,
  - backward difference formula (BDF): backward Euler method,
  - mesh adaptivity: *h*-version of DGM

Fundamentals of options (1)

- **option** - a special case of derivatives, contract between two parties about trading the asset at a certain future time

Basic option scheme

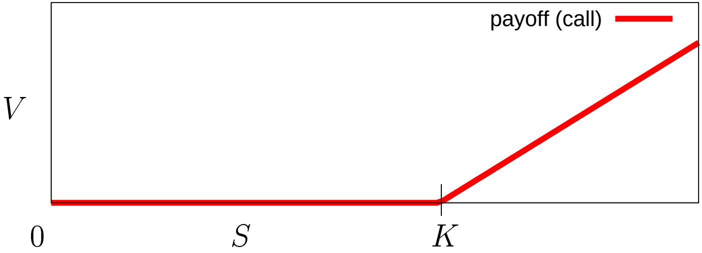
writer $\xrightarrow{\text{sells}}$ **premium** (market price) $\xrightarrow{\text{purchases}}$ **holder**

- **option** has a limited life time: matutity or expiration time T
- we distinguish:
  - **call** - gives the holder the right to *buy* the underlying for an agreed price K by the date T
  - **put** - gives the holder the right to *sell* the underlying for an agreed price K by the date T
  - **European** option - can be exercised only at expiration T
  - **American** option - can be exercised at any time $\tau \leq T$

Fundamentals of options (2)

- standard European type of options depending on a stock price $S = S(\tau) \Rightarrow V = V(S,t)$ value of certain type of option is given by underlying asset and time to expiration $t = T - \tau$
- terminal value of $V \Leftrightarrow$ payoff function

$$V_C^{Eu}(S) = \max(S - K, 0) \quad (\text{call}, \tau = T)$$



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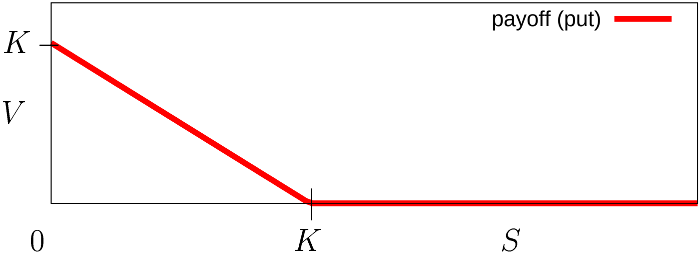
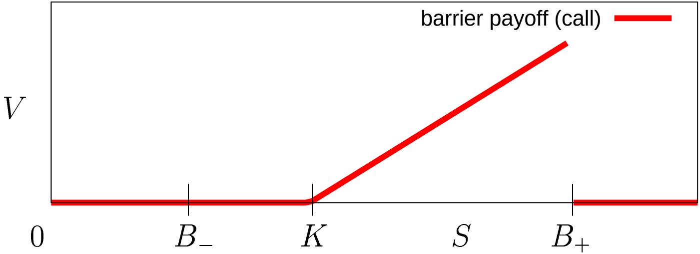
$$V_P^{Eu}(S) = \max(K - S, 0) \quad (\text{put}, \tau = T)$$



Fundamentals of options (3)

- we focus on double-barrier options: value $V(S)$ also depends on whether the price $S(\tau)$ lies between prescribed barriers B_- and B_+ during its lifetime
- discrete monitoring: barrier is active only at discrete time instances $M = \{0 = t_0^M < t_1^M < \ldots < t_{l-1}^M < t_l^M = T\}$
- terminal value of $V \Leftrightarrow$ payoff function

$$V_C^B(S) = V_C^{Eu}(S) \cdot \chi_{[B_-,B_+]} \quad (\text{call}, \tau = T)$$



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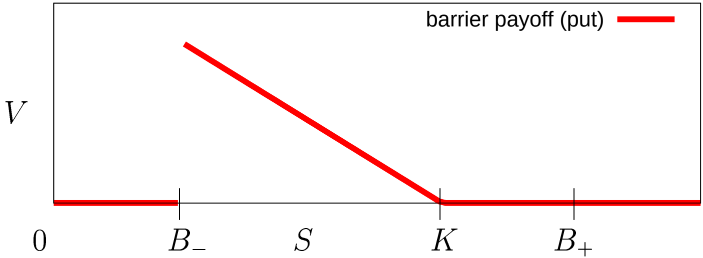


Black-Scholes (B-S) equation

- model of the classical financial market [Black, Scholes, Merton, 1973]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

$V = V(S, t; T, K, r, \sigma)$ - value of option
$t,\ 0 \le t \le T$ - time to expiration, $T - t$ - current time
S - price of stock at time t, i.e. $S = S(t)$, $0 \le S < +\infty$
$r,\ r > 0$ - risk-free interest rate
$\sigma,\ \sigma > 0$ - volatility of price S
K - strike price

- unsteady linear convection-diffusion-reaction equation

Scalar convection-diffusion-reaction equation

Let $\Omega = [0, S_{max}]$, $0 < B_- < B_+ < S_{max}$ and $T > 0$.

We seek $u : Q_T = \Omega \times (0, T) \to \boldsymbol{R}$ such that

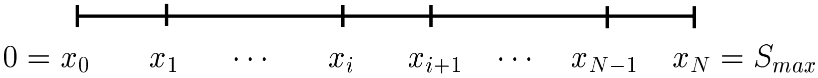
$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{L}_{BS}(u) &= 0 \quad \text{in } Q_T, \qquad (1) \\ u(0,t) = u_D^L(t) = 0 \quad \text{and} &\quad u(S_{max}, t) = u_D^U(t) = 0, \\ u(x,0) &= u^0(x) = V^B(x), \quad x \in \Omega, \ \text{(call/put)} \\ u(x, t_i^M) &= u(x, t_i^M -) \cdot \chi_{[B_-, B_+]}, \quad x \in \Omega, \ t_i \in M, \end{aligned}$$

where
$$\mathcal{L}_{BS}(u) = -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru, \quad r > 0, \sigma > 0$$

and $u^0 \in L^2(\Omega)$.

- reversal time $\Rightarrow$ initial condition is given by payoff function

Partitions



- let
$$0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = S_{max}$$
be a partition $\mathcal{T}_h, h > 0$ of $\overline{\Omega} = [0, S_{max}]$
- $\mathcal{T}_h = \{I_k, k = 0, \ldots, N-1\}$, $I_k = [x_k, x_{k+1}]$ are subintervals (elements) with length $h_k = x_{k+1} - x_k$
- we set a mesh size $$h := \max_{1 \le k \le N} (h_k)$$
- local quasi-uniformity
$$\exists C_q \ge 1 : \frac{h_k}{h_{k'}} \le C_q, \quad I_k, I_{k'} \text{ with common node}$$
- necessary cond.: $\exists k_1, k_2 \in \boldsymbol{N}$ such that $x_{k_1} = B_-$, $x_{k_2} = B_+$

Spaces of discontinuous functions

- let $p_k \geq 1$, $I_k \in \mathcal{T}_h$ be local polynomial degree,
- we set $\mathrm{p} \equiv \{p_k, I_k \in \mathcal{T}_h\} = \{p_k, k = 1, \ldots, N\}$
- over $\mathcal{T}_h$ we define:
  - *broken Sobolev space*

$$H^s(\Omega, \mathcal{T}_h) = \{v; v|_{I_k} \in H^s(I_k)\ \forall\, I_k \in \mathcal{T}_h\}, \quad s \geq 1$$

with the seminorm

$$|v|^2_{H^s(\Omega,\mathcal{T}_h)} \equiv \sum_{I_k \in \mathcal{T}_h} |v|^2_{H^s(I_k)} = \sum_{k=0}^{N-1} |v|^2_{H^s(I_k)}$$

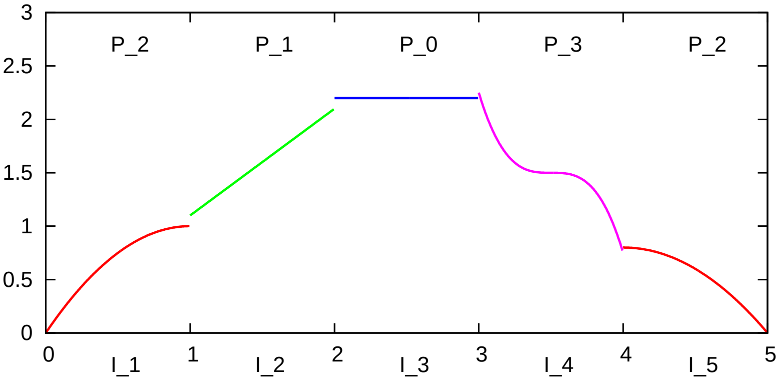
  - the space of piecewise polynomial functions

$$S_{hp} \equiv S_{h\mathrm{p}}(\Omega, \mathcal{T}_h) = \{v; v|_{I_k} \in P_{p_K}(I_k)\ \forall\, I_k \in \mathcal{T}_h\}$$

Example of a piecewise polynomial function

- obviously

$$\varphi \in S_h([0,5], \mathcal{T}_h) \subset H^1([0,5], \mathcal{T}_h)$$



Notation - mean value, jump

- for $v \in H^1(\Omega, \mathcal{T}_h)$ let us denote

$$v(x^+) = \lim_{\varepsilon \to 0+} v(x+\varepsilon) \quad \text{and} \quad v(x^-) = \lim_{\varepsilon \to 0+} v(x-\varepsilon)$$

- we define jump at endpoints of I_k

$$[v(x_k)] = v(x_k^-) - v(x_k^+) \quad \forall k = 1, \dots, N-1$$

- we define mean value at endpoints of I_k

$$\langle v(x_k) \rangle = \frac{1}{2}\left(v(x_k^-) + v(x_k^+)\right) \quad \forall k = 1, \dots, N-1$$

- we extend the definition for endpoints of domain $[0, S_{max}]$, i.e.

$$[v(x_0)] = -v(x_0^+), \ \langle v(x_0) \rangle = v(x_0^+), \ [v(x_N)] = v(x_N^-), \ \langle v(x_N) \rangle = v(x_N^-)$$

Space semi-discretization

- let u be a strong (regular) solution on Ω
- we multiply (1) by $v \in H^2(\Omega, \mathcal{T}_h)$,
- integrate over each $I_k \in \mathcal{T}_h$,
- apply Green's theorem,
- sum over all $I_k \in \mathcal{T}_h$,
- we include additional terms vanishing for regular solution,
- we obtain the identity

$$\left(\frac{\partial u}{\partial t}(t), v\right) + a_h^{\Theta}(u(t), v) + b_h(u(t), v) + (2r - \sigma^2)(u(t), v) + \alpha J_h^{\omega}(u(t), v) = \ell_h^{\Theta}(v)(t) \quad \forall v \in H^2(\Omega, \mathcal{T}_h)\ \forall t \in (0, T), \tag{2}$$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v) = \ell_h^{\Theta}(v)(t)$$

- time derivation, $u = u(t) \in H^2(\Omega, \mathcal{T}_h), t \in (0, T)$,

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v) = \ell_h^{\Theta}(v)(t)$$

- diffusion terms: $-\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} \rightsquigarrow \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right)$, $K(x) = \frac{1}{2}\sigma^2 x^2$,

$$\begin{aligned} a_h^{\Theta}(u,v) &= \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} \frac{1}{2}\sigma^2 x^2 \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x}\,\mathrm{d}x \\ &- \sum_{k=0}^{N} \left\langle \frac{1}{2}\sigma^2 x_k^2 \frac{\partial u}{\partial x} \right\rangle \cdot [v(x_k)] \\ &+ \Theta \sum_{k=0}^{N} \left\langle \frac{1}{2}\sigma^2 x_k^2 \frac{\partial v}{\partial x} \right\rangle \cdot [u(x_k)], \end{aligned}$$

- SIPG ($\Theta = -1$), NIPG ($\Theta = 1$), IIPG ($\Theta = 0$).

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^\Theta(u,v) + b_h(u,v) + \alpha J_h^\omega(u,v) + (2r - \sigma^2)(u,v) = \ell_h^\Theta(v)(t)$$

- convection terms: $-rx\frac{\partial u}{\partial x} \rightsquigarrow \frac{\partial}{\partial x} f(x,u),\ f(x,u) = (\sigma^2 - r)xu,$

$$\begin{aligned}
\widetilde{b}_h(u,v) &= -\sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} (\sigma^2 - r)xu \cdot \frac{\partial v}{\partial x}\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{N} \underbrace{(\sigma^2 - r)x_k u(x_k)}_{f(x_k, u(x_k))} \cdot [v(x_k)] \\
b_h(u,v) &= -\sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} (\sigma^2 - r)xu \cdot \frac{\partial v}{\partial x}\,\mathrm{d}x \\
&\quad + \sum_{k=0}^{N} H\left(u(x_k^-), u(x_k^+)\right) \cdot [v(x_k)]
\end{aligned}$$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v) = \ell_h^{\Theta}(v)(t)$$

- numerical flux:

$$H\Big(u(x^-), u(x^+)\Big) = \begin{cases} (\sigma^2 - r)x \cdot u(x^-), & \text{if } A > 0 \\ (\sigma^2 - r)x \cdot u(x^+), & \text{if } A \leq 0 \end{cases}$$

  where $A = (\sigma^2 - r)x$
- for $\Omega = [0, S_{max}]$ sign of A depends only on values σ^2 and r
- concept of upwinding

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v) = \ell_h^{\Theta}(v)(t)$$

- interior and boundary penalty:

$$J_h^{\omega}(u,v) = \sum_{k=0}^{N} \omega [u(x_k)]\,[v(x_k)]$$

- penalty parameter ω is weight function ($C_W > 0$)

$$\omega(x) = \frac{C_W}{d(x)}, \quad d(x) = \begin{cases} h_1/{p_1}^2 & , x = 0, \\ \min\left(\frac{h_k}{p_k^2}, \frac{h_{k+1}}{p_{k+1}^2}\right) & , x = x_k, k = 1, .., N-1 \\ h_N/{p_N}^2 & , x = S_{max}, \end{cases}$$

- α depends on the properties of function $K(x)$.

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\ominus}(u, v) + b_h(u, v) + \alpha J_h^{\omega}(u, v) + (2r - \sigma^2)(u, v) = \ell_h^{\ominus}(v)(t)$$

- reaction terms: $ru \rightsquigarrow (2r - \sigma^2)u$

$$(u, v) = \int_{\Omega} uv\mathrm{d}x = \sum_{k=0}^{N-1} \int_{x_k}^{x_{k+1}} uv\mathrm{d}x$$

- multiplicative factor $(2r - \sigma^2)$ results from Green's theorem,
- (broken) $L^2(\Omega)$-inner product.

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\Theta}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r-\sigma^2)(u,v) = \ell_h^{\Theta}(v)(t)$$

- right-hand-side $\leftrightarrow$ boundary conditions

$$\begin{aligned}\ell_h^{\Theta}(v)(t) = & -\Theta\frac{1}{2}\sigma^2 x_0^2 \cdot v'(x_0^+) \cdot u_D^L(t) \\ & + \Theta\frac{1}{2}\sigma^2 x_N^2 \cdot v'(x_N^-) \cdot u_D^U(t) \\ & + \alpha \cdot u_D^L(t) \cdot v(x_0^+) \\ & + \alpha \cdot u_D^U(t) \cdot v(x_N^-)\end{aligned}$$

- SIPG ($\Theta = -1$), NIPG ($\Theta = 1$), IIPG ($\Theta = 0$),
- barrier options $\rightarrow$ homogeneous b.c. $\rightarrow$ vanishing of $\ell_h^{\Theta}(v)(t)$

DG formulation

$$\left(\frac{\partial u}{\partial t}, v\right) + a_h^{\ominus}(u,v) + b_h(u,v) + \alpha J_h^{\omega}(u,v) + (2r - \sigma^2)(u,v) = 0$$

corresponds to (convection-diffusion-reaction) PDE

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(K(x)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x} f(u) + \beta u = 0 \quad \text{in } Q_T$$

with $K(x) = \frac{1}{2}\sigma^2 x^2$, $f(u) = (\sigma^2 - r)xu$ and $\beta = 2r - \sigma^2$

which is equivalent with original B-S equation

$$\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} - rx\frac{\partial u}{\partial x} + ru = 0 \quad \text{in } Q_T$$

Semi-discrete DG scheme

- $S_{hp} \subset H^2(\Omega, \mathcal{T}_h) \Rightarrow$ (2) makes sense for $u_h, v_h \in S_{hp}$
- we define new bilinear form

$$\mathcal{B}_h^\Theta(u, v) := a_h^\Theta(u, v) + b_h(u, v) + J_h^\omega(u, v) + (2r - \sigma^2)(u, v)$$

Semi-discrete DG solution

We say that u_h is a semi-discrete DG solution iff

$$\begin{aligned} &\text{a)} \quad u_h \in C^1(0, T; S_{hp}), \\ &\text{b)} \quad \left(\frac{\partial u_h(t)}{\partial t}, v_h\right) + \mathcal{B}_h^\Theta(u_h(t), v_h) = 0 \\ &\qquad\qquad \forall\, v_h \in S_{hp}, \;\; t \in (0, T) \\ &\text{c)} \quad u_h(0) = u_h^0, \; u_h^0 \text{ is } S_{hp}\text{-approximation of } u^0 \end{aligned} \tag{3}$$

- semi-discrete problem (3) represents ODEs,

Time discretization

- linearity of form $\mathcal{B}_h^{\Theta}(\cdot,\cdot) \Rightarrow$ implicit treatment,
- implicit approach via backward Euler method
- let $0 = t_0 < t_1 < \cdots < t_r = T$ be a partition of $(0, T)$, $\tau_l \equiv t_{l+1} - t_l,\ u_h(t_l) \approx u_h^l \in S_{hp}, \quad l = 0, \ldots, r$

First order implicit scheme

$$\frac{1}{\tau_l}\left(u_h^{l+1} - u_h^l, v_h\right) + \mathcal{B}_h^{\Theta}\left(u_h^{l+1}, v_h\right) = 0$$
$$\forall\, v_h \in S_{hp},\ l = 0, \ldots, r-1, \tag{4}$$

where u_h^0 is S_{hp}-approximation of u^0

- problem (4) $\Longleftrightarrow$ system of linear algebraic equations.

Mesh adaptation

- unsteady problem $\Rightarrow$ set up new triangulation $\mathcal{T}_{lh}$ at (each) time instance t_l,
- initial mesh $\mathcal{T}_{0h}$ - uniform very coarse grid,
- $\mathcal{T}_{lh}$ depends on $\mathcal{T}_{l-1,h}$ through h-adaptation operations:
  - (C) cutting elements: 1 split on 2,
  - (G) gluing elements: 2 joins into 1,
  - (M) movement of nodes (composition of (C) and (G))
- restriction on h-adaptation:
  - h_{min} - minimal admissible size of mesh step,
  - h_{max} - maximal admissible size of mesh step,
  - N_{max} - prescribed maximal number of elements,
  - keep local quasi uniformity,
  - keep nodes at prescribed barriers B_- and B_+.

Basic *h*-adaptation strategy

a) mesh refinement - in domains with irregular solution (low regularity) or with high value of residual estimate,

b) mesh coarsening - in domains with solution of high regularity and low value of residual estimate,

- regularity indicator based on shock capturing techniques from hyperbolic problems [Dolejší, Feistauer, 03]
- element-wise indicator jump indicator

$$g_{I_k}(u_h) \equiv \frac{\sum_{i=k}^{k+1} [u_h(x_k)]^2}{h_k^{2p_k+1}}, \quad k = 1, \dots, N-1,$$

- regions with smooth solution $u \Rightarrow g_{I_k} \approx 0$,
- regions with discontinuities of $u \Rightarrow g_{I_k} \gg 1$,
- cut-version of indicator $\widetilde{g_{I_k}}(u_h) : S_{hp} \rightarrow [0;1]$.

Residual estimator

- simple strong form of local (element-wise) residue

$$res_{I_k}(u_h) = \frac{\partial u_h}{\partial t} + \mathcal{L}_{BS}(u_h), \quad k = 1, \ldots, N-1$$

- local indicator $\eta_{I_k} = \|res_{I_k}(u_h)\|_{L^2(I_k)}$,
- global indicator (estimator)

$$\eta_G(u_h) = \sqrt{\sum_{k=1}^{N-1} \eta_{I_k}^2}$$

- stopping criterion for h-adaptivity:
  $\eta_G(u_h^l) \leq \varpi, \quad l = 1, \ldots, r$, where $\varpi > 0$ is given tolerance,
- requirement of uniform distribution of global residue: $\eta_{I_k} \leq \frac{\varpi}{N}$.

h-adaptive DGM algorithm

1. let $\varpi > 0$, $0 < h_{min} \leq h_{max}$ and N_{max} be given
2. $B_-, B_+ \longleftrightarrow \mathcal{T}_{0h}$ and S_{hp} be set up, let $u^0 \longleftrightarrow u_h^0$ be given
3. repeat time loop (until $t_l > T$) $(l = 1, \ldots, r)$

$$\left\{\begin{array}{ll}
\text{(a)} & \text{solve problem (4) on } \mathcal{T}_{l-1,h} \Rightarrow u_h^l \\
\text{(b)} & \text{evaluate } \widetilde{g_{I_k}}(u_h^l), \eta_{I_k}(u_h^l), I_k \in \mathcal{T}_{l-1,h} \Rightarrow \eta_G(u_h^l), \\
\text{(c)} & \text{if } \eta_G(u_h^l) > \varpi \Rightarrow h-\text{refinement} \\
 & \left\{\begin{array}{ll}
 \text{(ref1)} & h-\text{refine elements with } \eta_{I_k} > \frac{\varpi}{N} \\
 \text{(coa1)} & h-\text{derefine elements with } \eta_{I_k} < \delta\frac{\varpi}{N} \wedge \widetilde{g_{I_k}}(u_h^l) \approx 0 \\
 \text{(ref2)} & \text{construct new mesh } \mathcal{T}_{lh} \text{ and space } S_{hp}, \text{ go to (a)},
 \end{array}\right. \\
\text{(d)} & \text{if } \eta_G(u_h^l) \leq \frac{\varpi}{3} \Rightarrow h-\text{coarsening} \\
 & \left\{\begin{array}{ll}
 \text{(coa1)} & h-\text{derefine elements with } \eta_{I_k} < \delta\frac{\varpi}{N} \wedge \widetilde{g_{I_k}}(u_h^l) \approx 0 \\
 \text{(coa2)} & \text{construct new mesh } \mathcal{T}_{lh} \text{ and space } S_{hp}, \text{ go to (a)},
 \end{array}\right.
\end{array}\right.$$

Implementation aspects

- 1D FORTRAN 90 code with MATLAB post-processing
- restarted GMRES with block diagonal preconditioning,
- adaptive mesh step (h-version DG) and constant time step τ,
- all three IPG stabilizations: NIPG, SIPG and IIPG,
- piecewise $P_1 - P_3$ approximations,
- choice of value C_W in penalty parameter ω

	NIPG	IIPG	SIPG
P_1	1.0	5.0	10.0
P_2	1.0	10.0	40.0
P_3	1.0	20.0	100.0

Basis S_{hp}

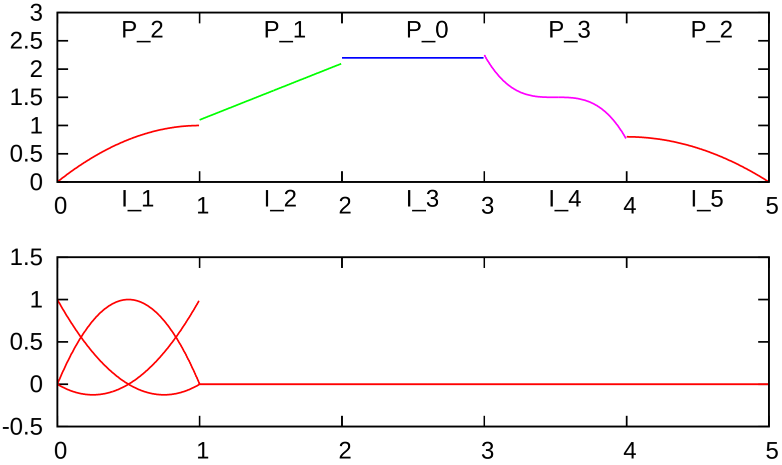
- local basis:
  $B_k = \{\varphi_{kj},\ \varphi_{kj} \in S_{hp},\ \text{supp}(\varphi_{kj}) \subset I_k,\ k = 1,..,dof_k\}$,
  $k = 1, \ldots, N-1$,
- global basis:
  $B = \{\varphi_{kj},\ \varphi_{kj} \in B_k,\ j = 1,..,dof_k,\ k = 1, \ldots, N-1\}$,

Linear algebraic representation

$$u_h^l(x) = \sum_{k=0}^{N-1} \sum_{j=1}^{dof_k} \xi_{lkj}\varphi_{kj},\ x \in \Omega,\ l = 0, 1, .., r,$$

$$u_h^l(x) \leftrightarrow \mathbf{U}_l = \{\xi_{lkj}\}_{kj} \in \boldsymbol{R}^{DOF},\ \ DOF = \sum_{k=1}^{N-1} dof_k.$$

Basis functions (fictional 1D problem)



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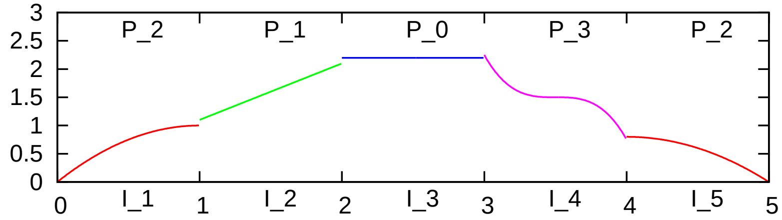

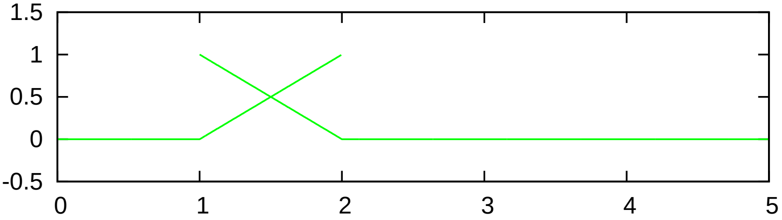

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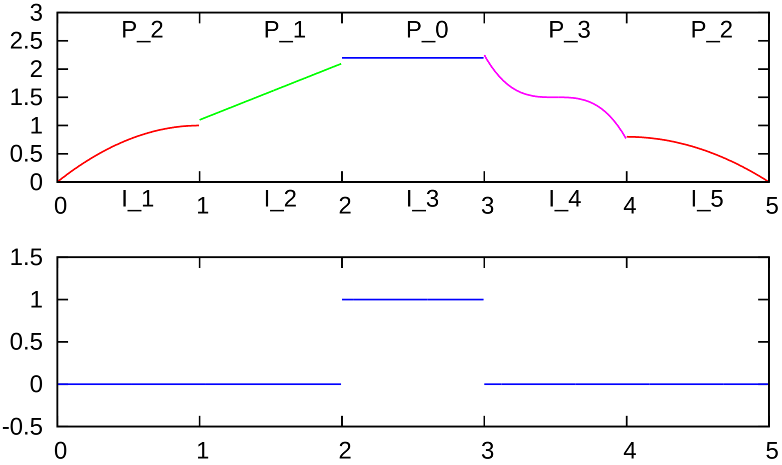
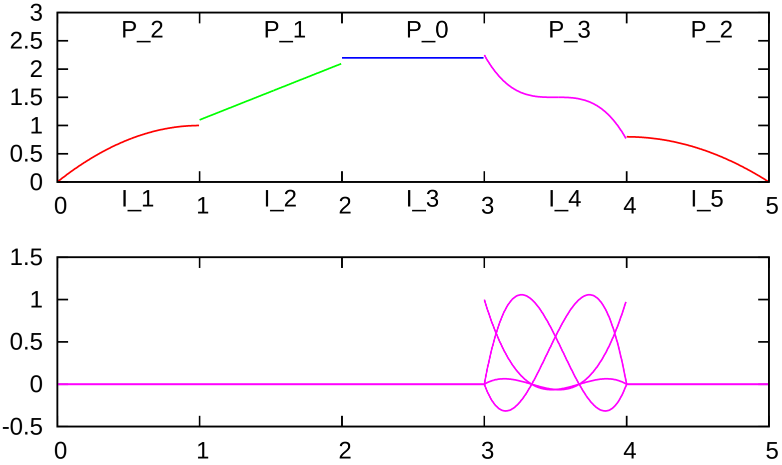
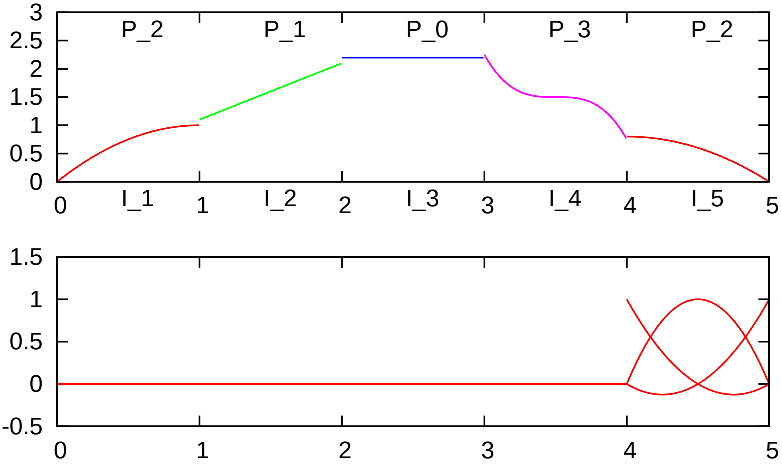


Basis functions (fictional 1D problem)



Basis functions (fictional 1D problem)



Linear algebraic problem

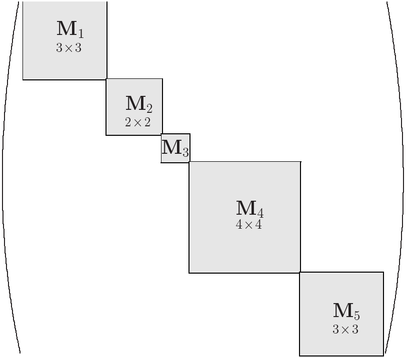
Backward Euler DG scheme

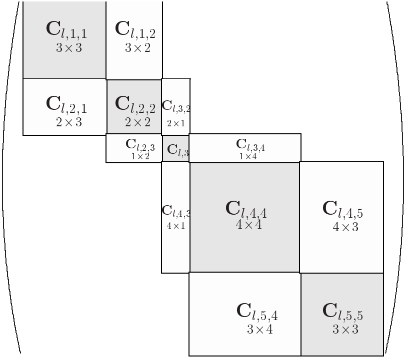
$$\Big((1 + \tau_l (2r - \sigma^2)) \mathbf{M}_l + \tau_l \mathbf{C}_l \Big) \mathbf{U}_{l+1} = \mathbf{q}_l, \; l = 0, .., r-1,$$

where

- $\mathbf{U}_l$ unknown vector,
- $\mathbf{M}_l$ mass matrix,
- $\mathbf{C}_l$ "flux" matrix representing convection, diffusion and penalty terms,
- $\mathbf{q}_l$ right-hand side (BC + terms from prev. time level),
- τ_l time step.

Matrix structure of **M** and $\mathbf{C}_l$ (fictional 1D problem)





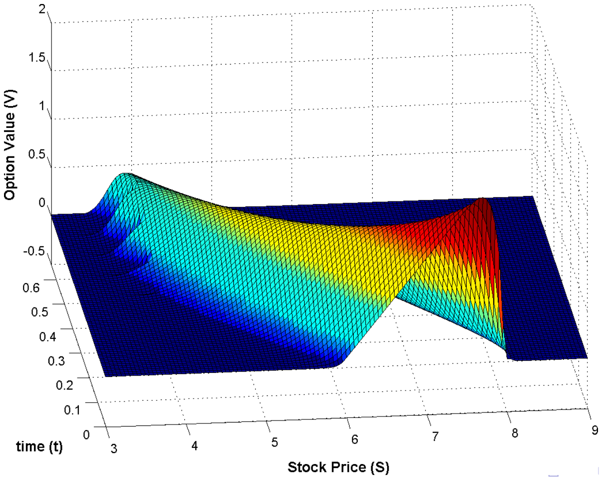
Example 1 - Double barrier knock-out call option (1)

- maximal stock price $S_{max} = 9 \rightarrow \overline{\Omega} = [0, 9]$,
- bottom barrier $B_{-} = 4.0$, upper barrier $B_{+} = 8.0$,
- strike price $K = 6.0$,
- risk-free interest rate $r = 1.y^{-1}$, volatility $\sigma^2 = 0.01y^{-1}$,
- initial uniform mesh with spatial step $h = 0.25$,
- next time instants: h-adaptation with parameters $h_{min} = 10^{-3}$, $h_{max} = 0.5$,
- constant time step $\tau = \frac{1}{120}$, piecewise P_1 approximation,
- final time $T = \frac{8}{12}$ y (i.e. 8 months),
- monthly monitoring: $t_i = \frac{i}{12}$, $i = 0, \ldots, 8$.

Development of option value and residual estimator

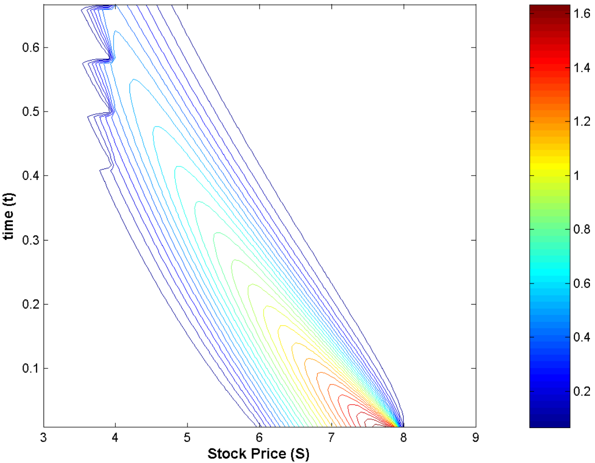
Example 1 - Double barrier knock-out call option (2)

- option value in space-time domain (3D plot and contours)



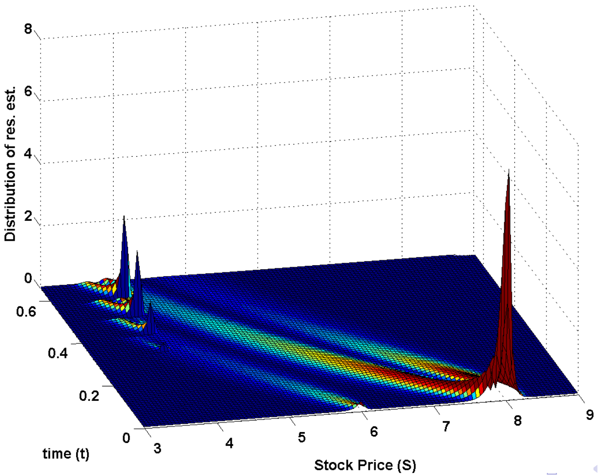
Example 1 - Double barrier knock-out call option (2)

- option value in space-time domain (3D plot and contours)



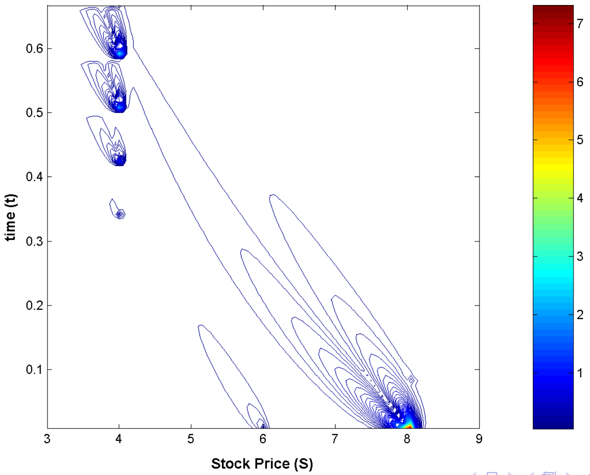
Example 1 - Double barrier knock-out call option (3)

- residual estimator in space-time domain (3D plot, contours)



Example 1 - Double barrier knock-out call option (3)

- residual estimator in space-time domain (3D plot, contours)



Example 1 - Double barrier knock-out call option (4)

- comparison of *h*-adaptive and *h*-uniform approach

	adapted mesh		uniform mesh	
time	res_G	$\#\mathcal{T}_{lh}$	res_G	$\#\mathcal{T}_{lh}$
0.000000*	19.960869	24	10.888578	120
0.041667	2.143362	160	2.703930	120
0.083333*	1.558519	212	1.839824	120
0.125000	1.309592	181	1.923517	120
0.166667*	1.498133	178	1.459563	120
0.208333	0.741951	136	0.780314	120
0.250000*	0.567553	99	0.559641	120
0.291667	0.614157	76	0.313876	120
0.333333*	0.615153	58	0.476494	120
0.375000	0.580115	55	0.250752	120
0.416667*	0.449979	51	0.379719	120
0.458333	0.660444	46	0.278150	120
0.500000*	0.572154	44	0.475957	120
0.541667	0.408049	77	0.473129	120
0.583333*	0.188052	71	0.1403946	120
0.625000	0.657242	64	0.608179	120
0.666667*	0.119596	58	0.124287	120

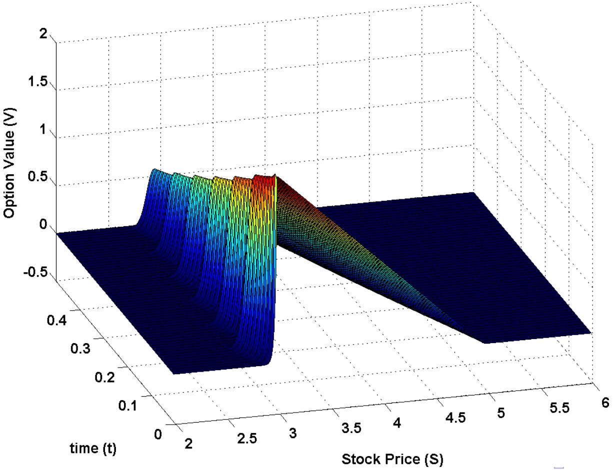
Example 2 - Double barrier knock-out put option (1)

- maximal stock price $S_{max} = 6 \rightarrow \overline{\Omega} = [0, 6]$,
- bottom barrier $B_{-} = 3.0$, upper barrier $B_{+} = 5.5$,
- strike price $K = 5$,
- risk-free interest rate $r = 0.7y^{-1}$, volatility $\sigma^2 = 10^{-6}y^{-1}$,
- initial uniform mesh with spatial step $h = 0.1 \rightarrow$
- next time instants: h-adaptation with parameters $h_{min} = 10^{-4}$, $h_{max} = 0.1$
- constant time step $\tau = \frac{1}{120}$, piecewise linear approximation,
- final time $T = \frac{6}{12}$ y (i.e. half year),
- monthly monitoring: $t_i = \frac{i}{12}$, $i = 0, \ldots, 6$.

Development of option value and residual estimator

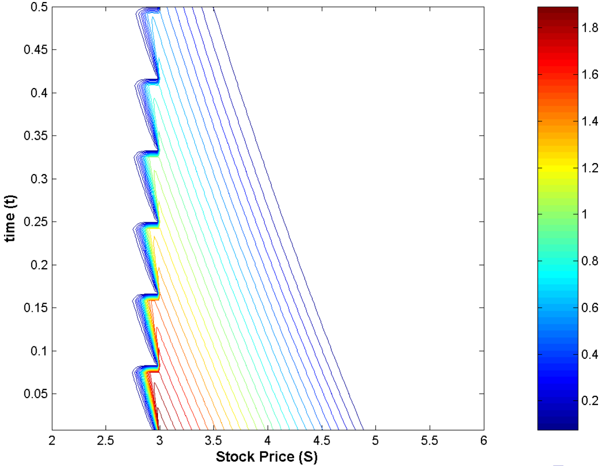
Example 2 - Double barrier knock-out put option (2)

- option value in space-time domain (3D plot and contours)



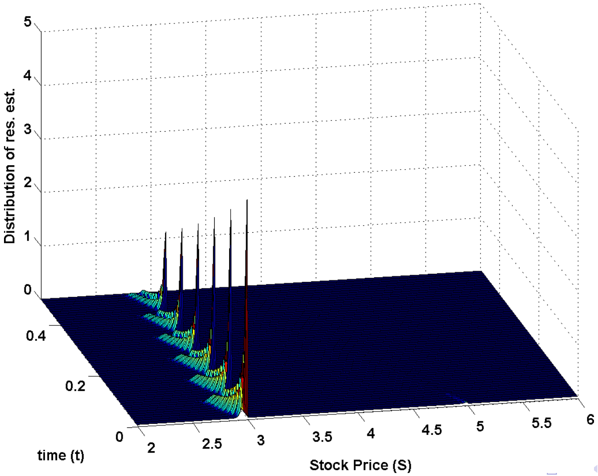
Example 2 - Double barrier knock-out put option (2)

- option value in space-time domain (3D plot and contours)



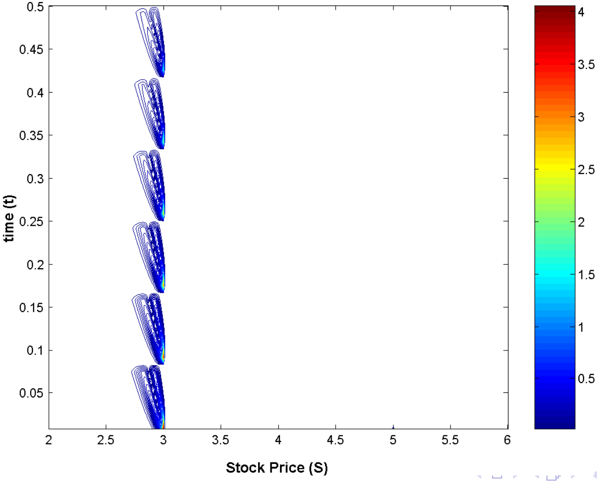
Example 2 - Double barrier knock-out put option (3)

- residual estimator in space-time domain (3D plot, contours)



Example 2 - Double barrier knock-out put option (3)

- residual estimator in space-time domain (3D plot, contours)



Example 2 - Double barrier knock-out put option (4)

- comparison of h-adaptive and h-uniform approach

	adapted mesh		uniform mesh	
time	res_G	$\#\mathcal{T}_{lh}$	res_G	$\#\mathcal{T}_{lh}$
0.000000*	18.066790	40	4.931157	400
0.041667	0.998231	175	1.075405	400
0.083333*	0.103857	176	0.207784	400
0.125000	0.446292	194	0.923517	400
0.166667*	0.074547	216	1.159563	400
0.208333	0.288194	258	0.780314	400
0.250000*	0.101703	220	0.157549	400
0.291667	0.267009	234	0.645247	400
0.333333*	0.088529	207	0.144814	400
0.375000	0.220599	209	0.517867	400
0.416667*	0.088126	196	0.695472	400
0.458333	0.179929	200	0.397773	400
0.500000*	0.070592	178	0.130634	400

Conclusion

Achieved results

- fundamentals of barrier option pricing,
- market model $\longleftrightarrow$ Black-Scholes equation (PDE),
- nonstationary linear convection-diffusion-reaction problem,
- DG space semi-discretization with backward Euler (implicit) time discretization,
- very simple mesh adaptivity technique

Future work

- extension to p-version of adaptivity,
- extension to space-time discontinuos Galerkin method,
- detailed investigation of cases $r \ll \sigma^2$ and $r \gg \sigma^2$,
- include other exotic options, e.g. lookback options

References

[AchP05] Y. Achdou and O. Pironneau. *Computational Methods for Option Pricing*. Frontiers in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, 2005.

[Sey08] R. Seydel. *Tools for Computational Finance: 4$^{\mathrm{th}}$ edition*. Springer, 2008.

Thank you for your attention