Error estimates for nonlinear convective problems in finite element methods

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1. Error estimates
   - Method of lines
   - Implicit scheme
2. From globally to locally Lipschitz **f**

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2. From globally to locally Lipschitz f

Scalar nonlinear convection

$$\text{a)} \quad \frac{\partial u}{\partial t} + \operatorname{div} \mathbf{f}(u) = g$$
$$\text{b)} \quad u\big|_{\Gamma_D \times (0,T)} = 0,$$
$$\text{c)} \quad u(x,0) = u^0(x), \quad x \in \Omega.$$

- $\mathbf{f} \in [C_b^2(\mathbb{R})]^d$,
- $\mathbf{f}'(u) \cdot \mathbf{n} \geq 0$ on Γ_N
- We assume u is sufficiently regular:

$$u, u_t \in L^2(0, T; H^{p+1}(\Omega))$$

- $p > \begin{cases} (d+1)/2, & \mathbf{f} \in [C_b^2(\mathbb{R})]^d, \\ (d-1)/2, & \mathbf{f} \in [C_b^3(\mathbb{R})]^d, \Gamma_N = \emptyset. \end{cases}$

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Definition

Standard conforming $p-$order FEM solution of the convection-diffusion problem:

$$\text{a) } u_h \in C^1([0,T]; V_h),$$

$$\text{b) } \left(\frac{\partial u_h(t)}{\partial t}, \varphi_h\right) + b(u_h(t), \varphi_h) = \ell(\varphi_h)(t), \quad \forall \varphi_h \in V_h,\ \forall t \in (0,T),$$

$$\text{c) } u_h(0) = u_h^0.$$

Convective term

$$b(u,v) = -\int_\Omega \mathbf{f}(u)\cdot\nabla v \,\mathrm{d}x + \int_{\Gamma_N} \mathbf{f}(u)\cdot\mathbf{n}\, v \,\mathrm{d}S$$

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c) $u_h(0) = u_h^0$.

Right-hand side term

$$\ell(v)(t) = \int_\Omega g(t) v \,\mathrm{d}x$$

Error estimates

- Let $e_h = \eta + \xi$, where $\eta = \Pi_h u - u$, $\xi = u_h - \Pi_h u \in V_h$.
- $\Pi_h : L^2(\Omega) \to V_h$ is the $L^2(\Omega)$–projection
- $\eta = O(h^\mu)$ in various norms, $\xi = ?$
- Subtract $eq(u) - eq(u_h)$, set $\varphi_h := \xi$

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$$\underbrace{\left(\frac{\mathrm{d}\xi}{\mathrm{d}t}, \xi\right)}_{\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi(t)\|^2} = b(u_h, \xi) - b(u, \xi) + \left(\frac{\mathrm{d}\eta}{\mathrm{d}t}, \xi\right)$$

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- For Gronwall we need only h^{2p+2}, $\|\xi\|^2$ on the RHS. Then

$$\max_{t\in[0,T]} \|\xi(t)\|^2 \mathrm{d}t = O(h^{2p+2}).$$

- Naively

$$b(u_h,\xi) - b(u,\xi) = \int_\Omega (\mathbf{f}(u) - \mathbf{f}(u_h)) \cdot \nabla\xi \,\mathrm{d}x \leq C\|e_h\||\xi|_1 \leq \frac{C}{\varepsilon}\|e_h\|^2 + \frac{1}{2}\varepsilon|\xi|_1^2,$$

- If we estimate using the inverse inequality

$$b(u_h,\xi) - b(u,\xi) \leq C\|e_h\||\xi|_1 \leq C\|e_h\| C_I h^{-1}\|\xi\|,$$

then we get $O\left(\exp\left(\frac{c}{h}\right) h^{2p+2}\right)$.

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The estimate of *Zhang, Shu (2004)*

Lemma

$$b(u_h,\xi) - b(u,\xi) \le C\left(1 + \frac{\|e_h\|_\infty}{h}\right)\left(h^{2p+1} + \|\xi\|^2\right)$$

- If $\mathbf{f} \in [C_b^3(\mathbb{R})]^d$, then we get a factor of $\frac{\|e_h\|_\infty^2}{h}$
- If $\|e_h(t)\|_\infty = O(h)$, then

$$b_h(u_h,\xi) - b_h(u,\xi) \le C\left(h^{2p+1} + \|\xi\|^2\right).$$

- For an explicit scheme, *Zhang, Shu (2004)* use induction:

$$\|e_h(t_n)\| = O(h^{p+1/2}) \Rightarrow \|e_h(t_{n+1})\|_\infty = O(h) \Rightarrow \|e_h(t_{n+1})\| = O(h^{p+1/2})$$

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If $\|e_h(\vartheta)\|_\infty = O(h)$ for all $\vartheta \in (0,t)$, then

$$\|e_h\|_{L^\infty(0,t;L^2(\Omega))} \leq C_T h^{p+1/2},$$

where C_T is independent of h, t.

Main theorem

Let $p > (d+1)/2$. Then

$$\|e_h\|_{L^\infty(L^2)} \leq C_T h^{p+1/2},$$

Proof:

- Nonlinear Gronwall-type lemma.
- Continuous mathematical induction (Y. R. Chao, 1919).

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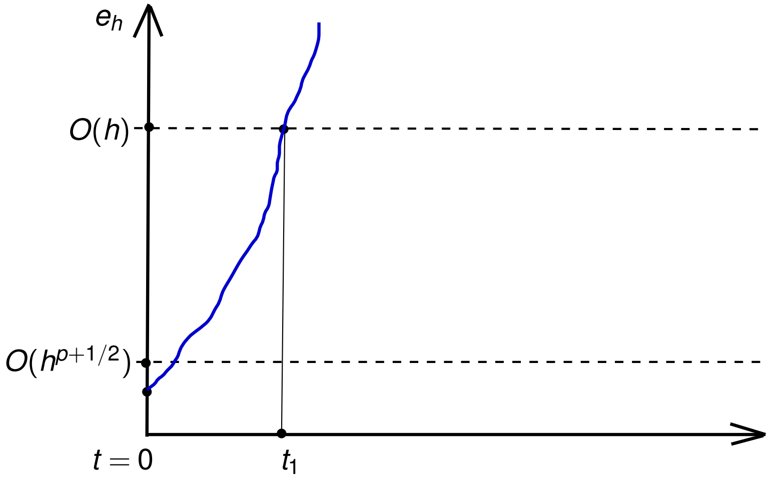
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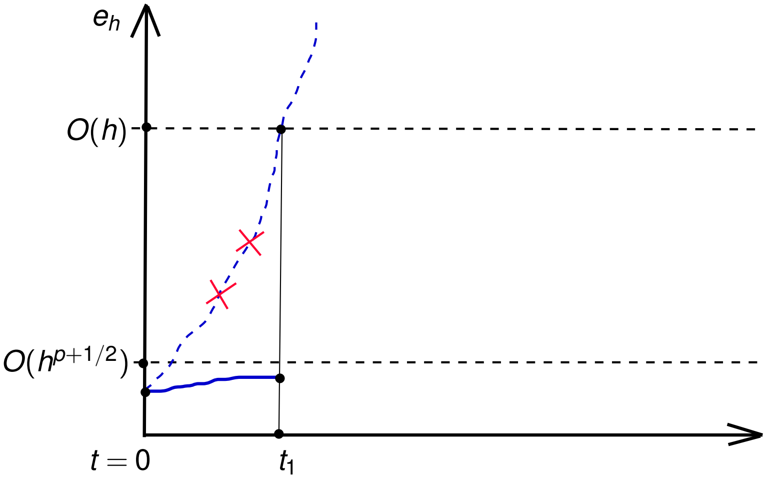
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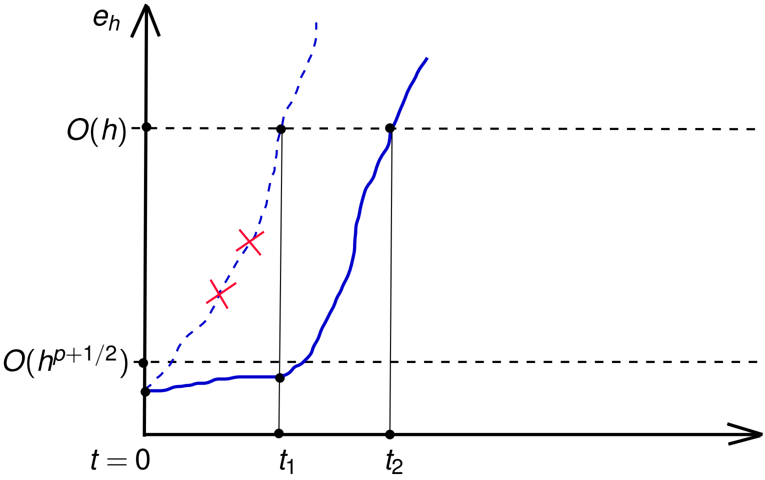
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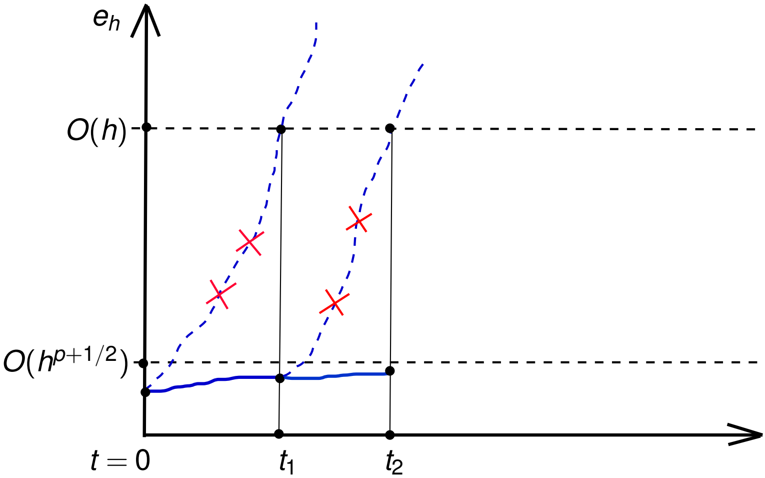
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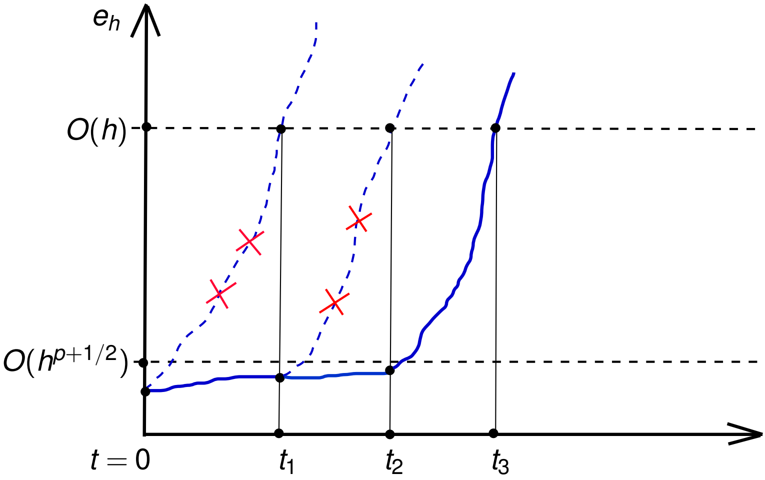
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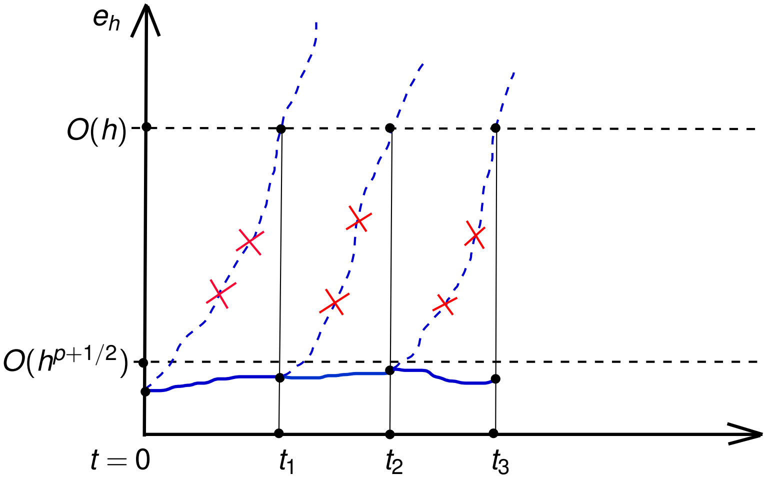
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Continuous (real) mathematical induction

Chao 1919

$\varphi(t)$ is a propositional function depending on $t \in [0, T]$ s.t.

$$(i) \quad \varphi(0) \text{ is true,}$$
$$(ii) \quad \exists \delta_0 > 0: \ \varphi(t) \text{ implies } \varphi(t+\delta), \ \forall t, \ \forall \delta \in [0, \delta_0].$$

Then $\varphi(t)$ holds for all $t \in [0, T]$.

Continuous (real) mathematical induction

Stronger version

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(iii) &\quad \forall t_1, t_2: \ \text{If } \varphi \text{ holds on } (t_1, t_2) \text{ then } \varphi(t_2) \text{ holds.}
\end{aligned}
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Proof of the key estimate

Lemma

$$b(u_h,\xi)-b(u,\xi) \le C\left(1+\frac{\|e_h(t)\|_\infty^2}{h^2}\right)\left(h^{2p+1}|u(t)|^2_{H^{p+1}(\Omega)}+\|\xi\|^2\right)$$

Proof:

$$b(u_h,\xi)-b(u,\xi)=\int_\Omega \left(\mathbf{f}(u)-\mathbf{f}(u_h)\right)\cdot\nabla\xi\,\mathrm{d}x.$$

The Taylor expansion gives us

$$\mathbf{f}(u)-\mathbf{f}(u_h)=\mathbf{f}'(u)\xi+\mathbf{f}'(u)\eta-\tfrac{1}{2}\mathbf{f}''_{u,u_h}e_h^2.$$

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Proof of the key estimate

$$b(u_h,\xi)-b(u,\xi)=\underbrace{\int_\Omega \mathbf{f}'(u)\xi\cdot\nabla\xi\,\mathrm{d}x}_{(1)}+\underbrace{\int_\Omega \mathbf{f}'(u)\eta\cdot\nabla\xi\,\mathrm{d}x}_{(2)}-\underbrace{\tfrac{1}{2}\int_\Omega \mathbf{f}''_{u,u_h}e_h^2\cdot\nabla\xi\,\mathrm{d}x}_{(3)}$$

$$(1)=-\tfrac{1}{2}\int_\Omega \mathrm{div}\big(\mathbf{f}'(u)\big)\xi^2\,\mathrm{d}x\le C\|\xi\|^2.$$

$$(2)\le Ch^{p+1}C_Ih^{-1}\|\xi\|\le Ch^{2p}+\|\xi\|^2.$$

$$(3)\le C\|e_h\|_\infty\|e_h\|C_Ih^{-1}\|\xi\|\le C\frac{\|e_h\|_\infty^2}{h^2}\big(Ch^{2p+2}+\|\xi\|^2\big)+\|\xi\|^2.$$

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Proof of the key estimate

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$$(1)=-\tfrac{1}{2}\int_\Omega \mathrm{div}\big(\mathbf{f}'(u)\big)\xi^2\,\mathrm{d}x\le C\|\xi\|^2.$$

$$(2)\le Ch^{p+1}C_I h^{-1}\|\xi\|\le Ch^{2p}+\|\xi\|^2.$$

$$(3)\le C\|e_h\|_\infty\|e_h\|C_I h^{-1}\|\xi\|\le C\frac{\|e_h\|_\infty^2}{h^2}\big(Ch^{2p+2}+\|\xi\|^2\big)+\|\xi\|^2.$$

1. Error estimates
   - Method of lines
   - Implicit scheme
2. From globally to locally Lipschitz **f**

Definition

Let $0 = t_0 < t_1 < \cdots < t_{N+1} = T$, $\tau_n := t_{n+1} - t_n$

$$
\begin{aligned}
&\text{a) } u_h^n \in V_h,\\
&\text{b) } \left(\frac{u_h^{n+1} - u_h^n}{\tau_n}, \varphi_h \right) + b(u_h^{n+1}, \varphi_h) = \ell(\varphi_h)(t_{n+1}),\\
&\qquad\qquad \forall\, \varphi_h \in V_h,\ \forall\, n = 0, \cdots N,\\
&\text{c) } u_h^0 \approx u(0).
\end{aligned}
$$

Standard approach

- $eq(u) - eq(u_h)$
- test by ξ
- estimate b, ℓ
- use Gronwall's inequality.

Theorem

There does not exist a Gronwall type lemma which could prove the desired error estimate only from the error equation tested by ξ and estimates of individual terms contained therein.

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Continuation

Auxiliary problem

Given $\tau \geq 0$ and $U_h \in V_h$, we seek $u_\tau \in V_h$ such that

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By setting $U_h := u_h^n, \tau := \tau_n$, then $u_\tau = u_h^{n+1}$.
By setting $U_h := u_h^n, \tau := 0$, then $u_\tau = u_h^n$.

Lemma (Existence, uniqueness and continuity)

Let $\tau = O(h)$, then $\exists! u_\tau \in V_h$ and $\|u_\tau\|$ depends continuously on τ.

Definition (Continuated discrete solution)

Let $\tilde{u}_h : [0, T] \to V_h$ be such that for $t \in [t_n, t_{n+1}]$ we define $\tilde{u}_h(t) := u_\tau$, the solution of the auxiliary problem with $\tau := t - t_n$ and $U_h := u_h^n$. Furthermore, we define $\tilde{e}_h := u - \tilde{u}_h$.

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$$\left(\frac{u_\tau - U_h}{\tau}, \varphi_h\right) + b(u_\tau, \varphi_h) = l(\varphi_h)(t), \quad \forall \varphi_h \in V_h.$$

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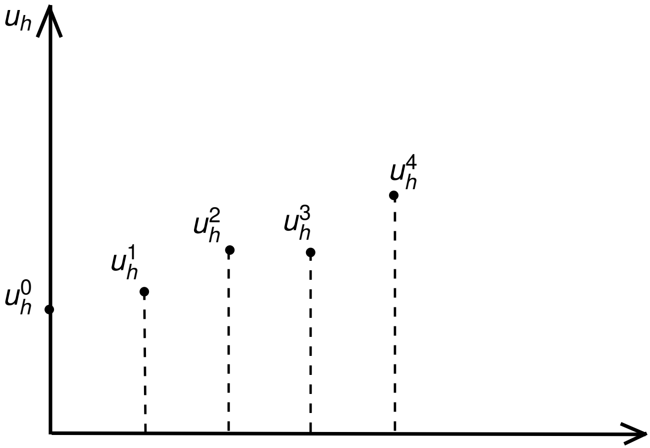

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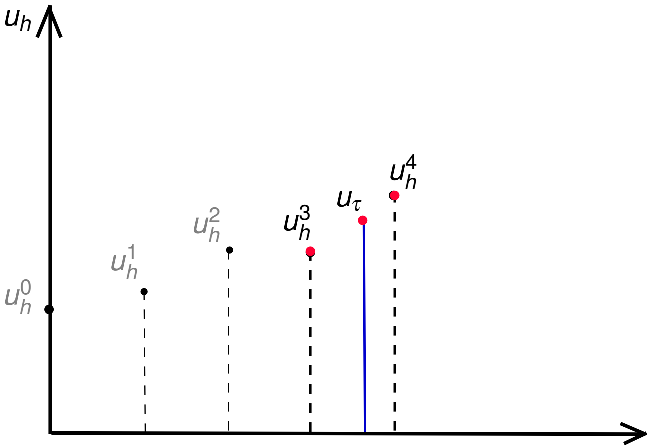

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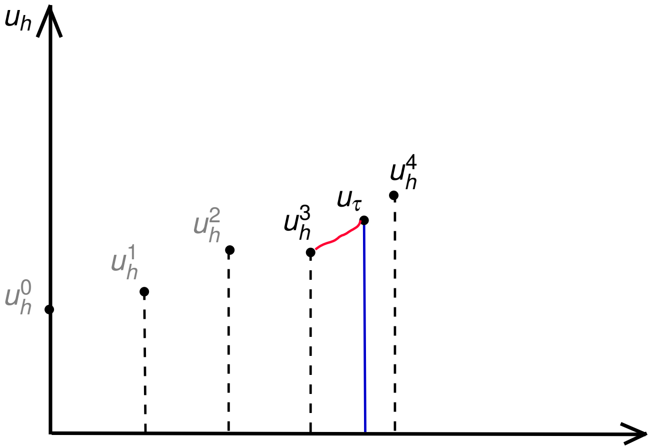

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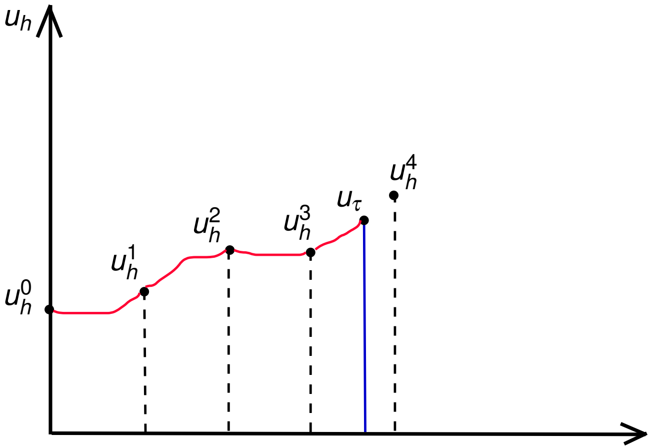

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Estimates for $\tilde{e}_h \quad \Longrightarrow \quad$ Estimates for e_h^n, $n = 0, \cdots, N+1$.

Lemma

If $\|\tilde{e}_h(\vartheta)\|_\infty = O(h)$ for all $\vartheta \in (0, t)$, then

$$\|\tilde{e}_h\|_{L^\infty(0,t;L^2(\Omega))} \leq C(h^{p+1/2} + \tau),$$

Main theorem

Let $p > (d+1)/2$ and $\tau = O(h^{1+d/2})$. Then

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- We assume only $\mathbf{f} \in (C^2(\mathbb{R}))^d$.
- Zhang & Shu modify $\mathbf{f}$ far from $\mathscr{R}(u)$ to obtain $\mathbf{f} \in (C_b^2(\mathbb{R}))^d$. This does not change u, but we get a completely new scheme.
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If $u_h(t) \in \mathscr{U}_h^{ad}(t)$, then

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Conclusions and outlook

- Estimates for nonlinear convection equations under high regularity assumptions.
- Analysis is valid for higher order elements: $p > (d-1)/2$.
- Unnatural CFL condition $\tau = O(h^{(1+d)/2})$ for implicit case.
- The situation would improve for higher order discretizations in time, e.g. BDF, space-time DG, ... CFL condition $\tau = O(h^{(1+d)/2k})$ for a k-th order scheme in time.
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Thank you for your attention.