SMOOTH APPROXIMATION OF DATA WITH APPLICATIONS TO INTERPOLATING AND SMOOTHING

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CONTENTS

PROBLEM OF APPROXIMATION

The usual problem is: We have a finite number of measured (sampled) function values f_j obtained at a finite number of nodes X_j but we are interested also in the intermediate values corresponding to other points.

Let us have N such (complex, in general) values $f_1, f_2, \ldots, f_N \in C$ measured at N mutually distinct nodes $X_1, X_2, \ldots, X_N \in R^n$.

Assume that $f_j = f(X_j)$ are measured values of some continuous function f while z is an approximating function to be constructed.

We put $n = 1$ in what follows.

INTERPOLATION

The condition for the construction of the approximating function z is

$$z(X_j) = f(X_j), \quad j = 1, \ldots, N.$$

Possible additional conditions: analogical equalities for the derivatives of z or minimization of some functionals applied to z.

There are many solutions z of different behavior between the nodes and different smoothness that satisfy the above equalities.

MORE GENERAL: SMOOTHING

Instead of satisfying the above equalities, we usually minimize the expression

$$\sum_{j=1}^{N} (z(X_j) - f_j)^2,$$

each term possibly multiplied by some weight (smoothing by the least squares method).
Additional conditions can be formulated, too.

SMOOTH APPROXIMATION

Let $\widetilde{W}$ be a linear vector space of complex functions g continuous together with their derivatives of all orders on the interval (a, b), which may be infinite. Let $\{B_l\}_{l=0}^{\infty}$ be a sequence of nonnegative numbers and let there be the smallest nonnegative integer L such that $B_L > 0$.

Put

$$(g, h)_L = \sum_{l=0}^{\infty} B_l \int_a^b [g^{(l)}(x)]^* h^{(l)}(x) \, \mathrm{d}x,$$

where * denotes the complex conjugate. Further put $|g|_L = \sqrt{(g, g)_L}$, i.e.

$$|g|_L^2 = \sum_{l=0}^{\infty} B_l \int_a^b |g^{(l)}(x)|^2 \, \mathrm{d}x.$$

SMOOTH APPROXIMATION

If the value of $|g|_L$ exists and is finite it is the *seminorm* of the function g. If the same is true also for the function h then $(g, h)_L$ has the properties of the *inner product* of g and h. The set of these functions forms the Hilbert space W corresponding to the sequence $\{B_l\}_{l=0}^{\infty}$.

If $B_0 > 0$ (i.e. $L = 0$) the expression $|g|_0 = \|g\|$ is the norm on W.

PROBLEM OF SMOOTH INTERPOLATION

Choose a system of functions $g_k \in W$, $k = 1, 2, \dots,$ that is complete and orthogonal (with respect to the inner product in W), i.e.,

$$(g_k, g_n)_L = 0 \quad \text{for} \quad k \neq n, \qquad (g_k, g_k)_L = |g_k|_L^2 > 0.$$

Put

$$z(x) = t(x) + \sum_{k=1}^{\infty} A_k g_k(x), \qquad |g_k|_L \neq 0, \qquad t(x) = \sum_{p=0}^{L-1} a_p \varphi_p(x),$$

where $\{\varphi_p\}$, $p = 0, 1, \dots, L-1$, is a set of mutually orthogonal functions from W such that

$$(\varphi_p, \varphi_q)_L = 0, \qquad p, q = 0, 1, \dots, L-1.$$

The set $\{\varphi_p\}$ is empty for $L = 0$.

PROBLEM OF SMOOTH INTERPOLATION

The problem of smooth interpolation is to find the coefficients of the expression

$$z(x) = \sum_{p=0}^{L-1} a_p \varphi_p(x) + \sum_{k=1}^{\infty} A_k g_k(x)$$

such that

$$z(X_j) = f_j, \quad j = 1, \ldots, N,$$

and

the quantity $|z|_L$ attains its minimum.

A. Talmi, G. Gilat: Method for smooth approximation of data. J. Comput. Phys. 23 (1977), 93–123.

PROBLEM OF SMOOTH INTERPOLATION

Put

$$R_L(x,y) = \sum_{k=1}^{\infty} \frac{g_k(x) g_k^*(y)}{|g_k|_L^2}.$$

Theorem. Let the series $R_L(x,y)$ converges for all $x, y \in (a,b)$. Then the problem of smooth approximation has the unique solution

$$z(x) = \sum_{p=0}^{L-1} a_p \varphi_p(x) + \sum_{j=1}^{N} \lambda_j R_L(x, X_j),$$

where the coefficients a_p and λ_j are the solution of the system of $N+L$ linear algebraic equations

$$\sum_{j=1}^{N} \lambda_j \varphi_p(X_j) = 0, \quad p = 0, 1, \ldots, L-1,$$

$$\sum_{p=0}^{L-1} a_p \varphi_p(X_i) + \sum_{j=1}^{N} \lambda_j R_L(X_i, X_j) = f_i, \quad i = 1, \ldots, N.$$

EXAMPLES OF BASIS FUNCTION SYSTEMS
COMPLEX EXPONENTIAL FUNCTIONS

Let the function f to be approximated be periodic, e.g. $f(x) = f(x + 2\pi)$. The possible choice of the basis functions is

$$g_k(x) = \exp(\,ikx), \quad k = \dots, -2, -1, 0, 1, 2, \dots .$$

This range of k requires a small change in the above formulae. It is easy to show that the system is complete and orthogonal with respect to the above introduced inner product $(g, h)_0$,

$$\|g_k\|^2 = 2\pi \sum_{l=0}^{\infty} B_l k^{2l}, \quad \text{and} \quad R_0(x, y) = \sum_{k=-\infty}^{\infty} \frac{\exp(\,ik(x-y))}{\|g_k\|^2}.$$

EXAMPLES OF BASIS FUNCTION SYSTEMS
I: TRANSFORMED COMPLEX EXPONENTIAL FUNCTIONS
Numerical results in graphs are denoted by the green dashed line

Let the function f to be approximated be nonperiodic on $(-\infty, \infty)$, $f^{(l)}(\pm\infty) = 0$ for all $l \geq 0$. The system of complex exponential functions is transformed into $(-\infty, \infty)$. By this passage, we also obtain

$$R_0(x,y) = \int_{k=-\infty}^{\infty} \frac{\exp(\,\mathrm{i}k(x-y))}{\|g_k\|^2}\,\mathrm{d}k.$$

Putting, in particular, $B_l = D^{2l}/(2l)!$, $0 < D < 1$, and denoting $r = |x-y|$, we obtain

$$R_0(x,y) = \frac{1}{2D\cosh(\pi r/(2D))}.$$

We use $D = \frac{1}{3}$ for numerical experiments.

EXAMPLES OF BASIS FUNCTION SYSTEMS II: ORTHONORMALIZED MONOMIALS

Numerical results in graphs are denoted by the black dotted line

Let the function f to be approximated be nonperiodic on $(-1, 1)$. The system of monomials

$$h_k(x) = x^k, \quad k = 0, 1, 2, \ldots,$$

is orthonormalized numerically on $(-1, 1)$ by the Gram-Schmidt procedure with respect to the inner product $(g, h)_0$. All computations, including the substitution in the series for $R_0(x, y)$, are carried out numerically.

We use $B_l = D^{2l}/(2l)!$, $D = \frac{1}{3}$ for numerical experiments.

K. Segeth: Smooth approximation and its application to some 1D problems. Proc. Conference Applications of Mathematics 2012. Prague, Institute of Mathematics of the AS CR 2012, 243–252.

EXAMPLES OF BASIS FUNCTION SYSTEMS III: TRANSFORMED COMPLEX EXPONENTIAL FUNCTIONS WITH A SPECIAL CHOICE OF B_l

Numerical results in graphs are denoted by the cyan dashed line

We consider the complex exponential functions transformed to $(-\infty, \infty)$. In the definition of inner product, we put $B_l = 0$ for all l with the exception of $B_2 = 1$. It means that we minimize the L^2 norm of the second derivative of the interpolant z.

We arrive at $R_2(x, y) = |x - y|^3$ and the function t has the form

$$t(x) = a_0 + a_1 x.$$

It is easy to find out that such a smooth approximation is, in fact, the well-known *cubic spline interpolation*.

COMPUTATIONAL COMPARISON WITH CLASSICAL INTERPOLATION

0: EXACT SOLUTION

Numerical results in graphs are denoted by the red solid line

IV: POLYNOMIAL INTERPOLATION

Numerical results in graphs are denoted by the blue dotted line

V: RATIONAL INTERPOLATION

Numerical results in graphs are denoted by the magenta dash-dot line

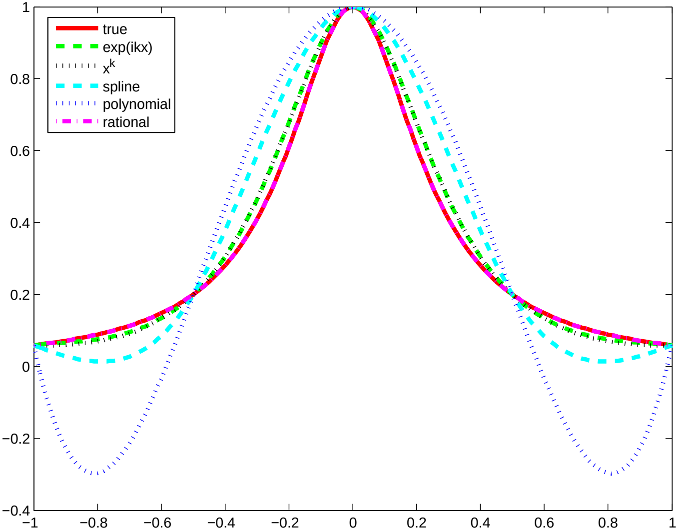
SMOOTH INTERPOLATION OF A SMOOTH FUNCTION

The function

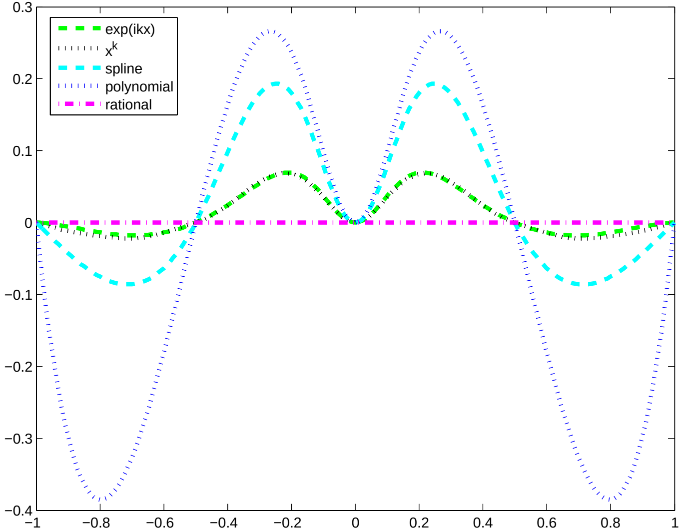
$$f(x) = \frac{1}{1 + 16x^2}$$

has "almost a pole" at $x = 0$. We constructed the smooth as well as classical interpolation in several both equidistant and nonequidistant grids.
The results for the equidistant grid with $N = 5$ follow. The rational interpolation is identical to the true function.

SMOOTH INTERPOLATION OF A SMOOTH FUNCTION RESULT



SMOOTH INTERPOLATION OF A SMOOTH FUNCTION
ERROR



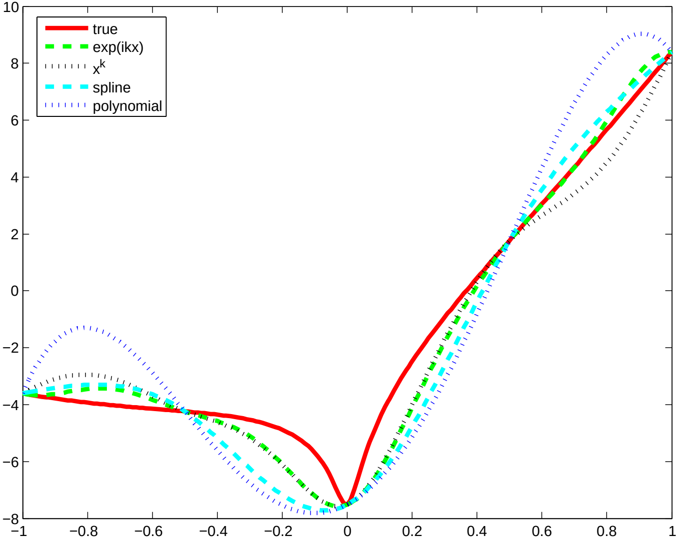
SMOOTH INTERPOLATION OF A NONSMOOTH FUNCTION

The function

$$f(x) = 3(x+1)^2 + \ln((\tfrac{1}{10}x)^2 + 10^{-5}) + 1$$

has “almost a singularity” at $x = 0$. We constructed the smooth as well as classical interpolation in several both equidistant and nonequidistant grids.
The results for the equidistant grid with $N = 5$ follow except for the very inaccurate results obtained by the rational interpolation.

SMOOTH INTERPOLATION OF A NONSMOOTH FUNCTION RESULT



SMOOTH INTERPOLATION OF A NONSMOOTH FUNCTION ERROR