A priori diffusion–uniform error estimates for singularly perturbed problems – DG and higher order time discretizations

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Continuous problem

- Let $\Omega \subset \mathbb{R}^d$ be polygonal domain, $Q_T \equiv \Omega \times (0, T)$, we seek $u : Q_T \to \mathbb{R}$ such that

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$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot F(u) &= \varepsilon \Delta u + g \quad \text{in } Q_T, \\ u\Big|_{\partial\Omega\times(0,T)} &= u_D, \\ u(x,0) &= u^0(x), \quad x \in \Omega, \end{aligned}$$

Notation

- elements K, $h_K = \text{diam}(K)$, $h = \max_K h_K$
- edges $\Gamma_h = \bigcup_K \partial K$
- arbitrary but fixed normals $\mathbf{n}$ to edges Γ_h
- $V_{h,p}$ be the space of piecewise polynomials up to degree p
- for $v \in V_{h,p}$, $x \in \Gamma_h$ we set $v_L(x) = \lim_{\delta\to 0+} v(x - \delta\mathbf{n})$ and $v_R = \lim_{\delta\to 0+} v(x + \delta\mathbf{n})$
- for $v \in V_{h,p}$, $x \in \Gamma_h$ we set $[v] = v_L - v_R$ and $\langle v \rangle = \frac{v_L + v_R}{2}$
- $\Pi_h : L^2(\Omega) \to V_{h,p}$ be L^2–orthogonal projection

Diffusive form A_h

$$A_h(u,w) = \sum_K \int_K \nabla u \cdot \nabla w \,\mathrm{d}x - \int_{\Gamma_h} \Big(\langle \nabla u \rangle \cdot \mathbf{n}[w] - \langle \nabla w \rangle \cdot \mathbf{n}[u] \Big) \,\mathrm{d}S + \int_{\Gamma_h} \sigma [u]\,[w] \,\mathrm{d}S,$$

Diffusive form A_h

- $A_h(v, w)$ be linear and nonsymmetric
- $|||v|||^2 = A_h(v, v)$
- $A_h(v, w) \leq C|||v|||\ |||w||| \quad \forall v, w \in V_{h,p}$
- $A_h(v - \Pi_h v, w) \leq Ch^p|v|_{H^{p+1}(\Omega)}|||w||| \quad \forall w \in V_{h,p}$

Convective form b_h

$$b_h(u,w) = \int_{\Gamma_h} H(u_L, u_R, \mathbf{n})\,[w]\,\mathrm{d}S - \sum_K \int_K F(u)\cdot\nabla w\,\mathrm{d}x,$$

Numerical fluxes

- $H(v, w, \mathbf{n})$ be Lipschitz continuous
- $H(v, v, \mathbf{n}) = F(v) \cdot \mathbf{n}$
- $H(v, w, \mathbf{n}) = -H(w, v, \mathbf{n})$

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- $H(v, w, \mathbf{n}) = -H(w, v, \mathbf{n})$
- $(H(v, w, \mathbf{n}) - F(q) \cdot \mathbf{n})(v - w) \geq 0 \quad \forall q \in [v, w]$

Convective form b_h

- $b_h(v, w)$ be nonlinear in v and linear in w
- $b_h(u, w) - b_h(v, w) \leq C\|u - v\| \, |||w||| \quad \forall u, v, w \in V_{h,p}$

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Lemma

Let $u \in H^{p+1}(\Omega)$, $U \in V_{h,p}$ and $\xi = U - \Pi_h u \in V_{h,p}$. Then there exists a constant C independent of h, such that

$$
\begin{aligned}
& b_h(u, \xi) - b_h(U, \xi) \\
& \leq C \left(1 + \frac{\|u - U\|^2_{L^\infty(\Omega)}}{h^2} \right) \left(h^{2p+1} |u|^2_{H^{p+1}(\Omega)} + \|\xi\|^2 \right)
\end{aligned}
$$

Source form ℓ_h

$$\ell_h(w)\,(t) = (g(t), w) + \varepsilon \int_{\partial\Omega} (-\nabla w \cdot \mathbf{n}\, u_D + \sigma u_D\, w)\ \mathrm{d}S.$$

Semi-discrete problem

- find $u_h \in C^1([0,T]; V_{h,p})$ such that

$$\left(\frac{\partial u_h}{\partial t}(t), v\right) + \varepsilon A_h(u_h(t), v) + b_h(u_h(t), v) = \ell_h(v)\,(t)$$
$$\forall\, v \in V_{h,p},\ \forall\, t \in [0,T],$$
$$(u_h(0), v) = (u^0, v) \quad \forall\, v \in V_{h,p}$$

Time discretization

- Let $t_m = m\tau \quad m = 0, \ldots, r$ be a partition of $[0, T]$ with a time step $\tau = T/r$,

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- let $u_h(t_m) = u_h^m \approx U^m \in V_{h,p}$ for $m = 0, \ldots, r$

Euler method

- Backward Euler method

$$(U^m - U^{m-1}, v) + \tau\varepsilon A_h(U^m, v) + \tau b_h(U^m, v) = \tau\ell(v)$$

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- continuation of U^m

$$(U_s - U^{m-1}, v) + s\varepsilon A_h(U_s, v) + sb_h(U_s, v) = s\ell(v)$$

Continued discrete solution – Euler method

- U_s is continuous with respect to s, $U_0 = U^{m-1}$, $U_\tau = U^m$
- If $\|u(t_{m-1}+s) - U_s\|_\infty \leq h$
  and $\|u(t_i) - U^i\|_\infty \leq h \; i = 0, \dots, m-1$
  then $\|u(t_{m-1}+s) - U_s\| \leq C(\tau + h^{p+1/2} + \varepsilon^{1/2} h^p)$
- If $\|u(t_{m-1}+s) - U_s\| \leq C(\tau + h^{p+1/2} + \varepsilon^{1/2} h^p)$,
  then $\|u(t_{m-1}+s+\delta) - U_{s+\delta}\|_\infty < h$
- assumption $C(\tau + h^{p+1/2} + \varepsilon^{1/2} h^p) < h^{1+d/2}$

- Midpoint rule

$$
\left(U^m - U^{m-1}, v\right) + \tau\varepsilon A_h\left(\frac{U^m + U^{m-1}}{2}, v\right)
$$

$$
+\tau b_h\left(\frac{U^m + U^{m-1}}{2}, v\right) = \tau\ell(v)
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- continuation of U^m

$$(U_s - U^{m-1}, v) + s\varepsilon A_h\left(\frac{U_s + U^{m-1}}{2}, v\right)$$
$$+s b_h\left(\frac{U_s + U^{m-1}}{2}, v\right) = s\ell(v)$$

Midpoint rule

$$b_h\left(u(t_{m-1}+\frac{s}{2}),v\right)-b_h\left(\frac{U_s+U^{m-1}}{2},v\right)$$

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$$b_h\left(u(t_{m-1}+\frac{s}{2}),v\right)-b_h\left(\frac{u(t_{m-1}+s)+u^{m-1}}{2},v\right)$$
$$\leq C\tau^2\|v\|$$
$$b_h\left(\frac{u(t_{m-1}+s)+u^{m-1}}{2},v\right)-b_h\left(\frac{U_s+U^{m-1}}{2},v\right)$$

Midpoint rule

Theorem

Let u be sufficiently smooth weak solution and U be its discrete solution defined by midpoint rule. Let $\tau \leq c \max(\varepsilon, h)$. Let $C(\tau^2 + h^{p+1/2} + \varepsilon^{1/2} h^p) < h^{1+d/2}$ ($p > 1 + d/2$). Then

$$\|u^m - U^m\| \leq C(\tau^2 + h^{p+1/2} + \varepsilon^{1/2} h^p).$$

BDF

- second order BDF

$$\left(\frac{3}{2}U^m - 2U^{m-1} + \frac{1}{2}U^{m-2}, v\right)$$
$$+\tau\varepsilon A_h(U^m, v) + \tau b_h(U^m, v) = \tau\ell(v)$$

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- continuation of U^m

$$\left(\frac{\tau+2s}{\tau+s}U_s - \frac{\tau+s}{\tau}U^{m-1} + \frac{s^2}{\tau^2+\tau s}U^{m-2}, v\right)$$
$$+s\varepsilon A_h(U_s, v) + sb_h(U_s, v) = s\ell(v)$$

BDF

Theorem

Let u be sufficiently smooth weak solution and U be its discrete solution defined by BDF. Let $\tau \leq c \max(\varepsilon, h)$. Let $C(\tau^2 + h^{p+1/2} + \varepsilon^{1/2} h^p) < h^{1+d/2}$ ($p > 1 + d/2$). Then

$$\|u^m - U^m\| \leq C(\tau^2 + h^{p+1/2} + \varepsilon^{1/2} h^p).$$

Time discretization

- Let $I_m = (t_{m-1}, t_m)$
- $V_h^\tau = \{v \in L^2(0, T, V_h) : v|_{I_m} \in P^q(I_m, V_h)\}$
- $v \in V_h^\tau$: $v_\pm^m = v(t_m\pm) = \lim_{t\to t_m\pm} v(t)$, $\{v\}_m = v_+^m - v_-^m$
- Radau quadrature on I_m:

$$\int_{t_{m-1}}^{s} f(t)dt \approx Q_s[f] = \sum_{i=0}^{q} w_i f(t_{m-1} + s\vartheta_i)$$

Time discontinuous Galerkin

- time discontinuous Galerkin: $U \in V_h^\tau$

$$\int_{I_m} (U', v) + \varepsilon A_h(U, v) + b_h(U, v) dt$$
$$+(\{U\}_{m-1}, v_+^{m-1}) = \int_{I_m} \ell(v) dt, \quad \forall v \in V_h^\tau$$

Time discontinuous Galerkin

- $\int_{t_{m-1}}^{s} b_h(u, v) - b_h(U_s, v) dt$
- $v = U_s - \Pi u \notin V_h^\tau$

Time discontinuous Galerkin – modification

- time discontinuous Galerkin – modification: $U \in V_h^\tau$

$$
\begin{aligned}
& Q_\tau \left[(U', v) + \varepsilon A_h(U, v) + b_h(U, v) \right] \\
& + (\{U\}_{m-1}, v_+^{m-1}) = Q_\tau[\ell(v)], \quad \forall v \in V_h^\tau
\end{aligned}
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Time discontinuous Galerkin – modification

- time discontinuous Galerkin – modification: $U \in V_h^\tau$

$$Q_\tau \left[(U', v) + \varepsilon A_h(U, v) + b_h(U, v)\right] + (\{U\}_{m-1}, v_+^{m-1}) = Q_\tau[\ell(v)], \quad \forall v \in V_h^\tau$$

- continuation of U: $U_s \in V_h^\tau$

$$Q_s \left[(U', v) + \varepsilon A_h(U, v) + b_h(U, v)\right] + (\{U\}_{m-1}, v_+^{m-1}) = Q_s[\ell(v)], \quad \forall v \in V_h^\tau$$

Time discontinuous Galerkin – modification

- $Q_s\left[b_h(u,v) - b_h(U_s,v)\right]$
- $v = U_s - \Pi u$

Semi–implicit Euler method

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Semi–implicit Euler method

$$b_h(u(t_{m-1} + s), v) - b_h(U^{m-1}, v) \tag{1}$$

Semi–implicit Euler method

$$b_h(u(t_{m-1}+s),v) - b_h(U^{m-1},v) \tag{1}$$

- $v = U^{m-1} - \Pi u^{m-1}$
- $v = U_s - \Pi u(t_{m-1}+s)$

Thank you for your attention.