SUPERAPPROXIMATION OF THE PARTIAL DERIVATIVES
IN THE SPACE OF LINEAR TRIANGULAR
AND BILINEAR QUADRILATERAL FINITE ELEMENTS

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Abstract
A method for the second-order approximation of the values of partial derivatives
of an arbitrary smooth function \( u = u(x_1, x_2) \) in the vertices of a conformal and
nonobtuse regular triangulation \( T_h \) consisting of triangles and convex quadrilaterals
is described and its accuracy is illustrated numerically. The method assumes that
the interpolant \( \Pi_h(u) \) in the finite element space of the linear triangular and bilinear
quadrilateral finite elements from \( T_h \) is known only.

1. Introduction
The problem to find second-order approximations of the first partial derivatives of
smooth functions \( u \) in the vertices of triangulations by means of the interpolant \( \Pi_h(u) \)
only is actual since its formulation in [6] in the year 1967. Besides the widely ac-
knowledged method [7] there exist successful methods like [5] and [3]. In this paper,
we generalize the method of averaging from [2] to nonobtuse regular triangulations
consisting of triangles as well as convex quadrilaterals in general. Numerical ex-
periments indicate the second-order accuracy of this procedure. These high-order
approximations of the partial derivatives have many applications. See [1] for some of
them.

We denote \([a_1, a_2]\) the Cartesian coordinates of a point \( a \) and \(|ab|\) the length of the
segment \( \overline{ab} \). For arbitrary points \( a^1, \ldots, a^n \), operations \(+\) and \(-\) mean addition
and subtraction modulo \( m \) on the set \( \{1, \ldots, m\} \).

2. Bilinear quadrilateral finite elements
Besides the linear triangular finite elements, we work with the following bilinear
quadrilateral ones.

Definition 1. A reference bilinear finite element consists of
Figure 1: The reference square.

a) the reference square $\hat{K} = \hat{a}^1\hat{a}^2\hat{a}^3\hat{a}^4$ from Fig. 1,
b) the local space $Q^{(1)} = \{ a + b\xi + c\eta + d\xi\eta \mid a, b, c, d \in \mathbb{R} \}$ and of
c) the parameters $\hat{p}(\hat{a}^1), \ldots, \hat{p}(\hat{a}^4)$ related to every function $\hat{p} \in Q^{(1)}$. The parameters determine the function $\hat{p}$ uniquely.

**Definition 2.** A bilinear quadrilateral finite element consists of
a) an image $K = a^1a^2a^3a^4$ of $\hat{K}$ by the injective bilinear mapping
\[
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = F_K(\xi, \eta) \equiv \sum_{i=1}^{4} \hat{N}^i(\xi, \eta) \begin{bmatrix} a^i_1 \\ a^i_2 \end{bmatrix}
\]
with the Lagrange base functions
\[
\hat{N}^1(\xi, \eta) = (1 - \xi)(1 - \eta)/4, \quad \hat{N}^2(\xi, \eta) = (1 + \xi)(1 - \eta)/4,
\hat{N}^3(\xi, \eta) = (1 + \xi)(1 + \eta)/4, \quad \hat{N}^4(\xi, \eta) = (1 - \xi)(1 + \eta)/4
\]
in the space $Q^{(1)}$ related to the nodes $\hat{a}^1, \ldots, \hat{a}^4$ consecutively. Then $F_K(\hat{a}^i) = a^i$ for $i = 1, \ldots, 4$ obviously and $F_K$ is an injection if and only if $K$ is a convex quadrilateral, i.e. the inner angle $\angle a^{i-1}a^ia^{i+1}$ of $K$ is less than $\pi$ for $i = 1, \ldots, 4$ due to [4], Section 3.3,
b) the local space $Q^{(1)}_K = \{ q \mid q = \hat{q} \circ F^{-1}_K \text{ for some } \hat{q} \in Q^{(1)} \}$ and of
c) the parameters $q(a^1), \ldots, q(a^4)$ related to every $q \in Q^{(1)}_K$. The parameters determine the function $q$ uniquely.

**Lemma 1.** The functions $1, x_1, x_2$ belong to $Q^{(1)}_K$ for every convex quadrilateral $K$.

Proof. If $K = a^1a^2a^3a^4$ is a convex quadrilateral then $Q^{(1)}_K = \{ q \mid q \circ F_K \in Q^{(1)} \}$ is a direct consequence of Definition 2. This and
\[
1 \circ F_K = 1 \in Q^{(1)}
\]
\[
x_1 \circ F_K = \hat{N}^1(\xi, \eta)a^1_1 + \ldots + \hat{N}^4(\xi, \eta)a^4_1 \in Q^{(1)}
\]
\[
x_2 \circ F_K = \hat{N}^1(\xi, \eta)a^1_2 + \ldots + \hat{N}^4(\xi, \eta)a^4_2 \in Q^{(1)}
\]
give us the statement.
Definition 3. If $K$ is a triangle and convex quadrilateral then we denote by $\Pi_K(u)$ the linear and bilinear interpolant of a function $u \in C(K)$ in the vertices of $K$, respectively.

Lemma 2. Let us consider a bilinear quadrilateral finite element $K = a^1a^2a^3a^4$, $l = 1, 2$ and a linear triangular finite element $T_j = a^{j-1}a^ja^{j+1}$. Then the graph of $\Pi_{T_j}(u)$ is the tangent plane to that of $\Pi_K(u)$ at the point $a^j$, so that

$$\frac{\partial \Pi_K(u)}{\partial x_l}(a^j) = \frac{\partial \Pi_{T_j}(u)}{\partial x_l} \forall u \in C(K)$$

for $j = 1, \ldots, 4$.

Proof. As the functions from $Q_K^{(1)}$ are linear on every side of $K$, $\Pi_K(u)$ is linear on the segments $a^{j-1}a^j$ and $a^ja^{j+1}$. Hence the segments $p^{j-1}p^j$ and $p^jp^{j+1}$ for $p^i = [a_i^1, a_i^2, u(a_i^3)]$, $i = j - 1, j, j + 1$, are subsets of graph($\Pi_K(u)$). These segments belong to a unique plane. This one is the tangent plane of graph($\Pi_K(u)$) at $a^j$ and it contains graph($\Pi_{T_j}(u)$) as well. Lemma 2 follows immediately.

3. Nonobtuse regular triangulations

The symbols $\mathbb{P}^{(1)}$ and $\mathbb{P}^{(2)}$ are reserved for the spaces of real linear and quadratic polynomials in two variables and $\Omega$ for a non-empty bounded connected polygonal domain in the plane. We say that $K$ is an element when $K$ is a triangle or a convex quadrilateral, denote $|K|$ the area of $K$, $h_K$ the diameter of $K$ and $\rho_K$ the maximal diameter of the circles inside of $K$.

A system $T_h$ of elements is said to be a triangulation of $\Omega$ when $\bigcup_{K \in T_h} K = \overline{\Omega}$, any two different elements have disjoint interiors and any side of an element is either a side of another element or a subset of the boundary $\partial \Omega$. Let us consider a vertex $a$ of (an element from) a triangulation $T_h$. We call $b$ a neighbour of $a$ (in $T_h$) when the segment $ab$ is a side of an element from $T_h$ and denote $\mathcal{N}_h(a)$ the set of neighbours of $a$ in $T_h$. We say that $a$ is an inner and boundary vertex when $a \in \Omega$ and $a \in \partial \Omega$, respectively.

Definition 4. A system $T$ of triangulations of $\Omega$ is said to be

a) a family when for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in T$ satisfying $h_K < \varepsilon$ for all $K \in \mathcal{T}_h$.

b) shape-regular when there is $\sigma > 0$ such that $\rho_K/h_K > \sigma$ for all elements $K$ of any triangulation from $T$.

We work with a shape-regular family $T$ of triangulations of $\Omega$ such that all inner angles of the triangles from any triangulation in $T$ are less than or equal to the right angle. We call these triangulations nonobtuse regular.
4. The method of averaging

It is well-known that $\frac{\partial u}{\partial x_l}(a) = \frac{\partial \Pi_K(u)}{\partial x_l}(a) + O(h_K)$ for a vertex $a$ of an element $K$ from a nonobtuse regular triangulation, function $u \in C^2(K)$ and for $l = 1, 2$. We construct a weight vector such that the corresponding weighted average of the values of $\frac{\partial \Pi_K(u)}{\partial x_l}$ in various vertices of the elements $K$ with vertex $a$ approximates $\frac{\partial u}{\partial x_l}(a)$ with an error of the second order. A special case of this construction has been analysed in [2] for the nonobtuse regular triangulations consisting of triangles only.

Calculating the approximations of $\frac{\partial u}{\partial x_l}(a)$, we use local Cartesian coordinates with origin $a$.

**Definition 5.** Let $\mathcal{T}_h$ be a nonobtuse regular triangulation. We say that $r = (b^1, \ldots, b^n)$ is a ring around

a) an inner vertex $a$ of $\mathcal{T}_h$ when

a1) $\{b^1, \ldots, b^n\} \supseteq \mathcal{N}_h(a)$ and

$$b^i \notin \mathcal{N}_h(a) \implies K = a b^{i-1} b^{i+1} \in \mathcal{T}_h \text{ and } \angle b^{i-1} a b^{i+1} > \pi/2,$$

a2) $\angle b^a b^1, \ldots, \angle b^{a-1} a b^n$ have the same orientation and

a3) $\angle b^a b^1 + \cdots + \angle b^{a-1} a b^n = 2\pi$.

b) a boundary vertex $a$ of $\mathcal{T}_h$ when there is an inner vertex $b^i$ such that

b1) $(b^1, \ldots, b^{i-1}, a, b^{i+1}, \ldots, b^n)$ is a ring around $b^i$ with $n \geq 5$ or

b2) $\overline{a b^{i-1} b^{i+1}} \in \mathcal{T}_h$ and $(b^1, \ldots, b^{i-1}, b^{i+1}, \ldots, b^n)$ is a ring around $b^i$.

We say that the triangles $U_1 = \overrightarrow{b^a b^1}, \ldots, U_n = \overrightarrow{b^n a b^n}$ are related to $r$ and set $H(a) = \max_{1 \leq i \leq n} |ab^i|$. 

![Figure 2](image-url) 

**Figure 2:** A ring around a) an inner vertex $a$ and b) a boundary one.

In Fig. 2, the thick lines denote the quadrilaterals from the given triangulation and the dotted lines indicate triangles $U_1, \ldots, U_6$ in the case a) and $U_1, \ldots, U_7$ in b).
**Definition 6.** Let \( l = 1, 2, r = (b^1, \ldots, b^n) \) be a ring around a vertex \( a \) of a nonobtuse regular triangulation and let \( u \in C(\overline{\Omega}) \). Then we set

\[
B_l[u](a) = f_1 \frac{\partial \Pi_1(u)}{\partial x_l} + \cdots + f_n \frac{\partial \Pi_n(u)}{\partial x_l},
\]

Here \( \Pi_1(u), \ldots, \Pi_n(u) \) are the linear interpolants of \( u \) in the vertices of the triangles \( U_1, \ldots, U_n \) related to \( r \) and the weight vector \( f = [f_1, \ldots, f_n]^T \) is the minimal 2-norm vector such that \( B_l[u](a) \) is consistent, i.e. \( B_l[u](a) = \partial u/\partial x_l(a) \) for all \( u \in \mathbb{P}^{(2)} \). Due to [2], \( f \) is the minimal 2-norm solution of the equations \( M(r)f = d \) with

\[
M(r) = \begin{bmatrix}
\frac{1}{D_1} x_n^2 y_1 - x_1^2 y_n & \frac{1}{D_1} x_1^2 y_2 - x_2^2 y_1 & \cdots & \frac{1}{D_1} x_{n-1}^2 y_n - x_n^2 y_{n-1} \\
\frac{y_n y_1 (x_n - x_1)}{D_1} & \frac{y_1 y_2 (x_1 - x_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (x_{n-1} - x_n)}{D_n} \\
\frac{y_n y_1 (y_n - y_1)}{D_1} & \frac{y_1 y_2 (y_1 - y_2)}{D_2} & \cdots & \frac{y_{n-1} y_n (y_{n-1} - y_n)}{D_n}
\end{bmatrix}, \quad d = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

\([x_i, y_i] = b^i \) and \( D_i = D(a, b^{-1}, b^i) \) for \( i = 1, \ldots, n \).

Definition 5 is in agreement with Lemma 2 and with the following statement:

**Lemma 3.** The system of equations \( M(r)f = d \) related to the ring \( r = (b^1, \ldots, b^4) \) around a vertex \( a \) is

a) unsolvable if \( a \) is a boundary vertex and

b) solvable if and only if the vertices \( b^1, a, b^3 \) as well as \( b^2, a, b^4 \) are situated on one straight-line if \( a \) is an inner vertex.

We omit the proof of Lemma 3.

**Example.** For \( a = [0, 0] \), we approximate the partial derivative \( \partial u/\partial x_1(a) = -0.5403023 \) of \( u(x_1, x_2) = \sin(1 + 2x_1 + x_2)/(x_2 - 2) \) by \( B_1[u](a) \). In Table 1, we use the ring from Fig. 2 a) with \( H(a) = 1.3453624/2^i \) for \( i = 1, \ldots, 8 \).

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<th>( i )</th>
<th>( H(a) )</th>
<th>( B_1<a href="a">u</a> )</th>
<th>( \partial u/\partial x_1(a) - B_1<a href="a">u</a> )</th>
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Table 1

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<th>$i$</th>
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<th>$\partial u/\partial x_1(a) - B_1<a href="a">u</a>$</th>
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Table 2

In Table 2, we use the ring from Fig. 2 b) with $H(a) = 2.3048861/2^i$ for $i = 1, \ldots, 8$.

This example indicates the second order of error of the approximations $B_i[u](a)$ both for the inner and the boundary vertices $a$, but an analysis of the accuracy of this averaging operator is necessary.

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References


