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AN EXTENSION OF SMALL-STRAIN MODELS TO THE LARGE-STRAIN RANGE BASED ON AN ADDITIVE DECOMPOSITION OF A LOGARITHMIC STRAIN

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Abstract

This paper describes model combining elasticity and plasticity coupled to isotropic damage. However, the the conventional theory fails after the loss of ellipticity of the governing differential equation. From the numerical point of view, loss of ellipticity is manifested by the pathological dependence of the results on the size and orientation of the finite elements. To avoid this undesired behavior, the model is regularized by an implicit gradient formulation. Finally, the constitutive model is extended to the large-strain regime. The large strain model is based on the additive decomposition of the logarithmic strain and preserves the structure of the small-strain theory.

1. Introduction

In this proceedings we will explore an extension of the small strain models combining elasticty and plasticity with isotropic damage, to the large strain regime. The extension to the large strain regime is based on the additive decomposition of the logarithmic strain into elastic and plastic part. The main acctractivness of this approach is in the modular framework consisting from three steps:

1. Definition of the elastic and plastic part of the logarithmic strain.
2. Computation of the generalized stress tensor, energy conjugated to the logarithmic strain and appropriate generalized stiffness via an algorithm that preserves structure of the small strain theory.
3. Transformation of the generalized tensors to the second Piola Kirchhoff stress and appropriate stiffness.

2. Constitutive model

In this section a model combining elasto-plasticity coupled with isotropic damage is described. The main feature of plasticity models is irreversibility of plastic strain while irreversible processes related to damage lead to degradation of stiffness. The basic equations include an additive decomposition of total strain into elastic (reversible) part and plastic (irreversible) part,

$$\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}, \tag{1}$$

the stress strain law,

$$\sigma_{ij} = (1 - \omega(\kappa))\,\bar{\sigma}_{ij} = (1 - \omega(\kappa))\,D^e_{ijkl}\varepsilon^e_{kl}, \tag{2}$$

loading-unloading conditions in Kuhn-Tucker form,

$$f(\bar{\sigma}_{ij}, \kappa) \leq 0 \qquad \dot{\lambda} \geq 0 \qquad \dot{\lambda} f(\bar{\sigma}_{ij}, \kappa) = 0, \tag{3}$$

evolution laws for plastic strain,

$$\dot{\varepsilon}^p_{ij} = \dot{\lambda}\frac{\partial f}{\partial \bar{\sigma}_{ij}}, \tag{4}$$

and for cumulated plastic strain,

$$\dot{\kappa} = \sqrt{\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij}}, \tag{5}$$

the law governing the evolution of the damage variable,

$$\omega(\kappa) = \omega_c(1 - \mathrm{e}^{-a\kappa}), \tag{6}$$

and the hardening law,

$$\sigma_Y(\kappa) = 1 + \sigma_H(1 - \mathrm{e}^{-s\kappa}). \tag{7}$$

In the equations above, $\bar{\sigma}_{ij}$ is the effective stress tensor, D^e_{ijkl} is the elastic stiffness tensor, f is the yield function, λ is the plastic multiplier, ω is the damage variable, κ is the cumulated plastic strain, σ_Y is the yield stress and s, a, σ_H and ω_c are positive material parameters, to be identified from experiments. Superior dot marks the derivative with respect to time. To describe specific material, suitable yield function needs to be introduced.

2.1. Regularization

Standard damage-plasticity models with softening may lead to localization of inelastic strain into narrow process zones. For traditional models formulated within the classical framework of continuum mechanics, such zones have an arbitrarily small thickness, and failure can occur at extremely low energy dissipation, which is not realistic. The mathematical model becomes ill-posed due to the loss of ellipticity of the governing differential equation and results obtained numerically are not objective with respect to the discretization. A general way to overcome pathological sensitivity of the numerical results to the finite element mesh is to adopt nonlocal continuum formulations. We focus our attention to the implicit gradient formulation, which requires only C^0 continuous finite element approximation. The nonlocal cumulated plastic strain is computed from a Helmholtz-type differential equation

$$\bar{\kappa} - l^2 \nabla^2 \bar{\kappa} = \kappa \tag{8}$$

with homogeneous Neumann boundary condition

$$\frac{\partial \bar{\kappa}}{\partial \mathbf{n}} = \mathbf{0}. \tag{9}$$

In (8), l is the length scale parameter and ∇ is the Laplace operator. Note that for present formulations, the nonlocal cumulated plastic strain affects only damage evolution while the yield condition remains local.

However, it can be shown that the implicit gradient formulation does not provide full regularization of the present model, thus the so-called over-nonlocal formulation has to be introduced. In this formulation, the damage variable is computed from over-nonlocal cumulated plastic strain, which is obtained as a combination of local cumulated plastic strain κ and nonlocal cumulated plastic strain $\bar{\kappa}$.

$$\hat{\kappa} = (1 - n)\kappa + n\bar{\kappa} \tag{10}$$

Full regularization can be achieved only if the parameter n is greater than 1.

3. Large-strain material models

Two sources of nonlinearities exist in the modeling of material. The first one is the material nonlinearity. A suitable material model at small strain has been presented in previous chapters. The second source of nonlinearity is related to the geometry. At first, we introduce strain measures. Next, extension of the constitutive model into the large-strain range based on the additive decomposition of the logarithmic strain is presented.

3.1. Generalized strain measures

A family of strain measures derived from the right Cauchy-Green deformation tensor was introduced by Seth and Hill [5, 6]. These generalized strain measures are defined as

$$\mathbf{E}^{(m)} = \frac{1}{2m} \left(\mathbf{C}^m - \mathbf{I} \right), \qquad m \neq 0 \tag{11}$$

$$\mathbf{E}^{(m)} = \frac{1}{2} \ln \mathbf{C}, \qquad m = 0 \tag{12}$$

where $\mathbf{I}$ is the second-order unit tensor. In the special cases when $m = 0$ and $m = 0.5$ we obtain the so-called Hencky (logarithmic) and Biot tensor, while for

$m = 1$ we obtain the right Green-Lagrange strain tensor. Recall that the Cauchy-Green deformation tensor is defined as

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{13}$$

where $\mathbf{F}$ is deformation gradient. The spectral decomposition of $\mathbf{C}$ is

$$\mathbf{C} = \sum_{a=1}^{3} \lambda_a \mathbf{N}^a \otimes \mathbf{N}^a \tag{14}$$

where λ_a are the eigenvalues of the right Cauchy-Green deformation tensor and $\mathbf{N}^a$ are the corresponding eigenvectors. Equations (11) and (12) can be rewritten as

$$\mathbf{E}^{(m)} = \frac{1}{2m}\left(\sum_{a=1}^{3}(\lambda_a^m - 1)\mathbf{N}^a \otimes \mathbf{N}^a\right), \qquad m \neq 0 \tag{15}$$

$$\mathbf{E}^{(m)} = \frac{1}{2}\sum_{a=1}^{3} \ln \lambda_a \mathbf{N}^a \otimes \mathbf{N}^a, \qquad m = 0 \tag{16}$$

For a hyperelastic material, the generalized stress tensors work-conjugate to the Seth-Hill strain measures and the corresponding generalized stiffness tensors can be derived from the Helmholtz free-energy density function ψ and expressed as

$$\mathbf{S}^{(m)} = \frac{\partial \psi}{\partial \mathbf{E}^{(m)}} \tag{17}$$

$$\mathbf{D}^{(m)} = \frac{\partial^2 \psi}{\partial \mathbf{E}^{(m)} \partial \mathbf{E}^{(m)}} \tag{18}$$

Application of the chain rule leads to the transformation formulas between generalized stress tensor and tangent moduli and Lagrangean objects: the second Piola-Kirchhoff stress

$$\mathbf{S} = 2\,\frac{\partial \psi}{\partial \mathbf{C}} = \mathbf{S}^{(m)} : \mathbf{P}^{(m)} \tag{19}$$

and the stiffness tensor

$$\mathbf{D} = \mathbf{P}^{(m)} : \mathbf{D}^{(m)} : \mathbf{P}^{(m)} + \mathbf{S}^{(m)} : \mathbf{L} \tag{20}$$

The transformation formulas exploit projection tensors

$$\mathbf{P}^{(m)} = 2\,\frac{\partial \mathbf{E}^{(m)}}{\partial \mathbf{C}} \tag{21}$$

$$\mathbf{L} = 4\,\frac{\partial^2 \mathbf{E}^{(m)}}{\partial \mathbf{C} \partial \mathbf{C}} \tag{22}$$

To differentiate the generalized strain measure with respect to the Cauchy-Green deformation tensor, we exploit the following formulas, which are valid if the eigenvalues are mutually different. The result for multiple eigenvalues is obtained by the rule of l'Hospital, see [2] for more details.

$$\frac{\partial \lambda_a}{\partial \mathbf{C}} = \mathbf{N}^a \otimes \mathbf{N}^a \tag{23}$$

$$\frac{\partial \mathbf{N}_a}{\partial \mathbf{C}} = \sum_{b \neq a}^{3} \frac{1}{\lambda_b - \lambda_a} \mathbf{N}^b \left(\mathbf{N}^a \otimes \mathbf{N}^b + \mathbf{N}^b \otimes \mathbf{N}^a \right) \tag{24}$$

Using equations (23) and (24) leads to the expression for the fourth-order projection tensor

$$\mathbf{P}^{(m)} = \sum_{a=1}^{3} \sum_{b=1}^{3} P_{aabb} \mathbf{N}^a \otimes \mathbf{N}^a \otimes \mathbf{N}^b \otimes \mathbf{N}^b + \sum_{a=1}^{3} \sum_{b \neq a}^{3} P_{abab} \left(\mathbf{N}^a \otimes \mathbf{N}^b \right) \otimes \left(\mathbf{N}^a \otimes \mathbf{N}^b + \mathbf{N}^b \otimes \mathbf{N}^a \right) \tag{25}$$

The components of this tensor are

$$P_{aabb} = \lambda_a^{m-1} \delta_{ab} \tag{26}$$

$$P_{abab} = \frac{\lambda_a^m - \lambda_b^m}{2m(\lambda_a - \lambda_b)} \tag{27}$$

where δ_{ab} is the Kronecker delta. The term $\mathbf{S}^{(m)} : \mathbf{L}$ is not described here; its detailed derivation can be found in [3] or [4].

3.2. Large-strain plasticity based on the logarithmic strain

The main attractiveness of the large-strain plasticity theory based on the logarithmic strain is in the modular framework consisting of three steps. In the first step, a logarithmic strain measure is computed from equation (12). In the second step, this strain measure enters a constitutive law, which may have an identical structure as in the small-strain theory. In the third step, the generalized stress tensor is transformed into the second Piola-Kirchhoff stress using expression (19) and the appropriate stiffness tensor is obtained merely by replacing the generalized stiffness tensor in equation (20) by the generalized algorithmic elasto-plastic stiffness tensor. We can define the elastic part of the logarithmic strain as

$$\mathbf{E}_e^{(0)} = \mathbf{E}^{(0)} - \mathbf{E}_p^{(0)} \tag{28}$$

with $\mathbf{E}^{(0)} = \frac{1}{2} \ln \mathbf{C}$, $\mathbf{E}_p^{(0)} = \frac{1}{2} \ln \mathbf{G}_p$. $\mathbf{G}_p$ is a Lagrangian object often called plastic metric. However, free energy function defined in terms of the logarithmic strain is not polyconvex. Polyconvexity of the free-energy function is a very important mathematical condition, which guarantee existence of the solution, see [11] for more details. Nevertheless, the model is suitable for description of materials for which the yield limit is reached at small strains.

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