ERROR ESTIMATES FOR NONLINEAR CONVECTIVE PROBLEMS IN THE FINITE ELEMENT METHOD

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Abstract

We describe the basic ideas needed to obtain apriori error estimates for a nonlinear convection diffusion equation discretized by higher order conforming finite elements. For simplicity of presentation, we derive the key estimates under simplified assumptions, e.g. Dirichlet-only boundary conditions. The resulting error estimate is obtained using continuous mathematical induction for the space semi-discrete scheme.

1. Continuous problem

Let \( \Omega \subset \mathbb{R}^d, d \in \mathbb{N} \), be a bounded open polyhedral domain. We treat the following nonlinear convective problem. Find \( u : \Omega \times (0, T) \to \mathbb{R} \) such that

\[
\begin{align*}
\text{(a)} & \quad \frac{\partial u}{\partial t} + \text{div} f(u) = g \quad \text{in } \Omega \times (0, T), \\
\text{(b)} & \quad u \big|_{\partial \Omega \times (0, T)} = 0, \\
\text{(c)} & \quad u(x, 0) = u^0(x), \quad x \in \Omega.
\end{align*}
\]

Here \( g : \Omega \times (0, T) \to \mathbb{R} \) and \( u^0 : \Omega \to \mathbb{R} \) are given functions. We assume that the \textit{convective fluxes} \( f = (f_1, \ldots, f_d) \in (C^2_b(\mathbb{R}))^d = (C^2(\mathbb{R}) \cap W^{2,\infty}(\mathbb{R}))^d \), hence \( f \) and \( f' = (f_1', \ldots, f_d') \) are \textit{globally Lipschitz continuous}.

By \((\cdot, \cdot)\) we denote the standard \( L^2(\Omega) \)-scalar product and by \( \| \cdot \| \) the \( L^2(\Omega) \)-norm. By \( \| \cdot \|_{\infty} \), we denote the \( L^\infty(\Omega) \)-norm. For simplicity of notation, we shall drop the argument \( \Omega \) in Sobolev norms, e.g. \( \| \cdot \|_{H^{p+1}} \) denotes the \( H^{p+1}(\Omega) \)-norm. We shall also denote the Bochner norms over the whole interval \([0, T]\) in concise form, e.g. \( \| u \|_{L^\infty(H^{p+1})} \) denotes the \( L^\infty(0, T; H^{p+1}(\Omega)) \)-norm.

2. Discretization

Let \( \mathcal{T}_h \) be a triangulation of \( \overline{\Omega} \), i.e. a partition into a finite number of closed simplexes with mutually disjoint interiors. We assume standard conforming properties: two neighboring elements from \( \mathcal{T}_h \) share an entire face, edge or vertex. We set \( h = \max_{K \in \mathcal{T}_h} \text{diam}(K) \).
Lemma 1. There exists a constant \( C_1 > 0 \) independent of \( h \) s.t. for all \( v_h \in S_h \)
\[
|v_h|_{H^1} \leq C_1 h^{-1} \|v_h\|,
\]
\[
\|v_h\|_\infty \leq C_1 h^{-d/2} \|v_h\|.
\]

We consider a system \( \{T_h\}_{h \in (0,h_0)} \), \( h_0 > 0 \), of triangulations of the domain \( \Omega \) which are shape regular and satisfy the inverse assumption, cf. [2]. Let \( p \geq 1 \) be an integer. The approximate solution will be sought in the space of globally continuous piecewise polynomial functions \( S_h = \{ v \in C(\bar{\Omega}); v|_{\Gamma_p} = 0 \text{ or } v|_{K} \in P^p(K) \forall K \in T_h \} \), where \( P^p(K) \) denotes the space of polynomials on \( K \) of degree \( \leq p \).

We discretize the continuous problem in a standard way. Multiply (1) by a test function \( \varphi_h \in S_h \), integrate over \( \Omega \) and apply Green’s theorem.

**Definition 1.** We say that \( u_h \in C^1([0,T];S_h) \) is the space-semidiscretized finite element solution of problem (1)-(3), if
\[
\frac{d}{dt}(u_h(t), \varphi_h) + b(u_h(t), \varphi_h) = l(\varphi_h)(t), \quad \forall \varphi_h \in S_h, \quad t \in (0,T).
\]

Here, we have introduced an approximation \( u_h^0 \in S_h \) of the initial condition \( u^0 \) and the convective and right-hand side forms defined for \( v, \varphi \in H^1(\Omega) \):
\[
b(v, \varphi) = -\int_{\Omega} f(v) \cdot \nabla \varphi \, dx, \quad l(\varphi)(t) = \int_{\Omega} g(t) \varphi \, dx.
\]

We note that a sufficiently regular exact solution \( u \) of problem (1) satisfies
\[
\frac{d}{dt}(u(t), \varphi_h) + b(u(t), \varphi_h) = l(\varphi_h)(t), \quad \forall \varphi_h \in S_h, \forall t \in (0,T),
\]
which implies the Galerkin orthogonality property of the error.

3. Key estimates of the convective terms

As usual in apriori error analysis, we assume that the weak solution \( u \) is sufficiently regular, namely
\[
u, u_t \in L^2(0,T;H^{p+1}(\Omega)), \quad u \in L^\infty(0,T;W^{1,\infty}(\Omega)),
\]
where \( u_t := \frac{\partial u}{\partial t} \). For \( v \in L^2(\Omega) \) we denote by \( \Pi_h v \) the \( L^2(\Omega) \)-projection of \( v \) on \( S_h \):
\[
\Pi_h v \in S_h, \quad (\Pi_h v - v, \varphi_h) = 0, \quad \forall \varphi_h \in S_h.
\]
Let \( \eta_h(t) = u(t) - \Pi_h u(t) \in H^{p+1}(\Omega) \) and \( \xi_h(t) = \Pi_h u(t) - u_h(t) \in S_h \) for \( t \in (0,T) \). Then we can write the error \( e_h \) as \( e_h(t) := u(t) - u_h(t) = \eta_h(t) + \xi_h(t) \). By \( C \) we denote a generic constant independent of \( h \), which may have different values in different parts of the text. Also, for simplicity of notation, we shall usually omit the argument \( t \) and subscript \( h \) in \( \xi_h(t) \) and \( \eta_h(t) \). In our analysis, we shall need the following standard inverse inequalities and approximation properties of \( \eta \), (cf. [2]):

**Lemma 1.** There exists a constant \( C_1 > 0 \) independent of \( h \) s.t. for all \( v_h \in S_h \)
\[
|v_h|_{H^1} \leq C_1 h^{-1} \|v_h\|,
\]
\[
\|v_h\|_\infty \leq C_1 h^{-d/2} \|v_h\|.
\]
Lemma 2. There exists a constant $C > 0$ independent of $h$ s.t. for all $h \in (0, h_0)$

\[
\|\eta_h(t)\| \leq C h^{p+1} |u(t)|_{H^{p+1}}, \\
\|\frac{\partial \eta_h(t)}{\partial t}\| \leq C h^{p+1} \frac{|\partial u(t)|}{h} |_{H^{p+1}}, \\
\|\eta_h(t)\|_\infty \leq C h |u(t)|_{W^{1, \infty}}.
\]

Lemma 3. There exists a constant $C \geq 0$ independent of $h, t$, such that

\[
b(u_h(t), \xi(t)) - b(u(t), \xi(t)) \leq C \left(1 + \frac{\|\varepsilon_h(t)\|_\infty}{h} \right) h^{2p+2} |u(t)|^2_{H^{p+1}} + \|\xi(t)\|^2.
\]  

Proof. The proof follows the arguments of [5], where similar estimates are derived for periodic boundary conditions or compactly supported solutions in 1D. The proof for mixed Dirichlet-Neumann boundary conditions is contained in [4]. We write

\[
b(u_h, \xi) - b(u, \xi) = \int_\Omega \left( f(u) - f(u_h) \right) \cdot \nabla \xi \, dx.
\]  

By the Taylor expansion of $f$ with respect to $u$, we have

\[
f(u) - f(u_h) = f'(u) \xi + f''(u) \xi \eta - \frac{1}{2} f''_{u,u_h} \varepsilon_h^2,
\]

where $f''_{u,u_h}$ is the Lagrange form of the remainder of the Taylor expansion, i.e. $f''_{u,u_h}(x, t)$ has components $f''_{u}(\vartheta_s(x, t) u(x, t) + (1 - \vartheta_s(x, t)) u_h(x, t))$ for some $\vartheta_s(x, t) \in [0, 1]$ and $s = 1, \ldots, d$. Substituting (9) into (8), we obtain

\[
b(u_h, \xi) - b(u, \xi) = \int_\Omega f'(u) \xi \cdot \nabla \xi \, dx + \int_{Y_1} f'(u) \xi \cdot \nabla \xi \, dx + \int_{Y_2} f'(u) \eta \cdot \nabla \xi \, dx + \frac{1}{2} \int_{Y_3} f''_{u,u_h} \varepsilon_h^2 \cdot \nabla \xi \, dx.
\]

We shall estimate these terms individually.

(A) Term $Y_1$: Due to Green’s theorem and the boundedness of $f''$ and the regularity of $u$, we have

\[
\int_\Omega f'(u) \xi \cdot \nabla \xi \, dx = -\frac{1}{2} \int_\Omega \text{div}(f'(u)) \xi^2 \, dx \leq C \|\xi\|^2.
\]

(B) Term $Y_2$: We define $\Pi_h^1 : (L^2(\Omega))^d \to (S_h^1)^d = \{ v \in (C(\Omega))^d ; v|_{\Gamma_D} = 0, v|_K \in (P^1(K))^d, \forall K \in T_h \}$, the $(L^2(\Omega))^d$-projection onto the space of continuous piecewise linear vector functions. From standard approximation results (similar to those of Lemma 2, cf. [2]), we obtain

\[
\|f'(u) - \Pi_h^1(f'(u))\|_\infty \leq C h |f'(u)|_{W^{1, \infty}} \leq C h \|f''\|_{L^\infty(\mathbb{R})}|u|_{L^\infty(W^{1, \infty})} = \tilde{C} h.
\]
Furthermore, due to the definition of \( \eta \), we have \( \int_{\Omega} \Pi_h^1(f'(u)) \cdot \nabla \eta \, dx = 0 \), since \( \Pi_h^1(f'(u)) \cdot \nabla \xi \in S_h \). Therefore, by Lemmas 1, 2 and Young’s inequality

\[
|Y_2| = \left| \int_{\Omega} (f'(u) - \Pi_h^1(f'(u))) \cdot \nabla \xi \, dx \right| \leq \| f'(u) - \Pi_h^1(f'(u)) \|_\infty C h^{-1} \| \xi \| \| \eta \|
\]

\[
\leq C h C h^{-1} \| \xi \| \| \eta \| \leq \| \xi \|^2 + C h^{2p+2} |u(t)|^2_{H^{p+1}}.
\]

(C) Term \( Y_3 \): We apply Lemmas 1, 2 and Young’s inequality:

\[
|Y_3| \leq C \| e_h \|_\infty \| e_h \| C h^{-1} \| \xi \| \leq C h^{-1} \| e_h \|_\infty (C h^{2p+2} |u(t)|^2_{H^{p+1}} + \| \xi \|^2).
\]

\[\square\]

4. Error analysis of the semidiscrete scheme

We proceed similarly as for a parabolic equation. By Galerkin orthogonality, we subtract (5) and (4) and set \( \varphi_h := \xi_h(t) \in S_h \). Since \( \left( \frac{\partial}{\partial t}, \xi_h \right) = \frac{1}{2} \frac{d}{dt} \| \xi_h \|^2 \), we get

\[
\frac{1}{2} \frac{d}{dt} \| \xi_h(t) \|^2 = b(u_h(t), \xi_h(t)) - b(u(t), \xi_h(t)) - \left( \frac{\partial \xi_h(t)}{\partial t}, \xi_h(t) \right).
\]

For the last right-hand side term, we use the Cauchy and Young’s inequalities and Lemma 2 and Lemma 3 for the convective terms. We integrate from 0 to \( t \in [0, T] \),

\[
\| \xi_h(t) \|^2 \leq C \int_0^t \left( 1 + \frac{\| e_h(\varphi) \|_\infty}{h} \right) \left( h^{2p+1} |u(\varphi)|^2_{H^{p+1}} + h^{2p+2} |u_t(\varphi)|^2_{H^{p+1}} + \| \xi_h(\varphi) \|^2 \right) \, d\varphi, \quad (11)
\]

where \( C \geq 0 \) is independent of \( h, t \). For simplicity, we have assumed that \( \xi_h(0) = 0 \), i.e. \( u_h^0 = \Pi_h u^0 \). Otherwise we must assume e.g. \( \| \xi_h(0) \|^2 \leq C h^{2p+1} |u^0|^2_{H^{p+1}} \) and include this term in the estimate.

We notice that if we knew \textit{apriori} that \( \| e_h \|_\infty = O(h) \) then the unpleasant term \( h^{-1} \| e_h \|_\infty \) in (11) would be \( O(1) \). Thus we could simply apply the standard Gronwall lemma to obtain the desired error estimates. We state this formally:

**Lemma 4.** Let \( t \in [0, T] \) and \( p \geq d/2 \). If \( \| e_h(\varphi) \| \leq h^{1+d/2} \) for all \( \varphi \in [0, t] \), then there exists a constant \( C_T \) independent of \( h, t \) such that

\[
\max_{\varphi \in [0,t]} \| e_h(\varphi) \|^2 \leq C_T h^{2p+1}.
\]

**Proof.** The assumptions imply, by the inverse inequality and estimates of \( \eta \), that

\[
\| e_h(\varphi) \|_\infty \leq \| \eta_h(\varphi) \|_\infty + \| \xi_h(\varphi) \|_\infty \leq C h \| u(t) \|_{W^{1,\infty}} + C h^{-d/2} \| \xi_h(\varphi) \|
\]

\[
\leq C h + C h^{1-d/2} \| e_h(\varphi) \| + C h^{-d/2} \| \eta_h(\varphi) \| \leq C h + C h^{p+1-d/2} \| u(\varphi) \|_{H^{p+1}(\Omega)} \leq C h,
\]

where the constant \( C \) is independent of \( h, \varphi, t \). Using this estimate in (11) gives us

\[
\| \xi_h(t) \|^2 \leq \tilde{C} h^{2p+1} + C \int_0^t \| \xi_h(\varphi) \|^2 \, d\varphi,
\]

(14)
where the constants $\bar{C}, C$ are independent of $h, t$. Gronwall’s inequality applied to (14) states that there exists a constant $\bar{C}_T$, independent of $h, t$, such that

$$\max_{\vartheta \in [0,t]} \|\xi_h(\vartheta)\|^2 + \frac{1}{2} \int_0^t |\xi_h(\vartheta)|_{\Gamma^1}^2 \, d\vartheta \leq \bar{C}_T h^{2p+1},$$

which along with similar estimates for $\eta$ gives us (12). \qed

Now it remains to get rid of the *apriori* assumption $\|e_h\|_\infty = O(h)$. In [5] this is done for an explicit scheme using mathematical induction. Starting from $\|e_h^0\| = O(h^{p+1/2})$, the following induction step is proved:

$$\|e_h^0\| = O(h^{p+1/2}) \implies \|e_h^{n+1}\| = O(h) \implies \|e_h^{n+1}\| = O(h^{p+1/2}). \quad (15)$$

For the method of lines we have continuous time and hence cannot use mathematical induction straightforwardly. However, we can divide $[0,T]$ into a finite number of sufficiently small intervals $[t_n, t_{n+1}]$ on which “$e_h$ does not change too much” and use induction with respect to $n$. This is essentially a *continuous mathematical induction* argument, a concept introduced in [1], which has many generalizations, cf. [3].

**Lemma 5** (Continuous mathematical induction). Let $\varphi(t)$ be a propositional function depending on $t \in [0,T]$ such that

(i) $\varphi(0)$ is true,
(ii) $\exists \delta_0 > 0 : \varphi(t)$ implies $\varphi(t+\delta), \, \forall t \in [0,T] \, \forall \delta \in [0,\delta_0] : t + \delta \in [0,T].$

Then $\varphi(t)$ holds for all $t \in [0,T].$

**Remark 1** Due to the regularity assumptions, the functions $u(\cdot), u_h(\cdot)$ are continuous mappings from $[0,T]$ to $L^2(\Omega)$. Since $[0,T]$ is a compact set, $e_h(\cdot)$ is a *uniformly continuous* function from $[0,T]$ to $L^2(\Omega)$. By definition,

$$\forall \epsilon > 0 \exists \delta > 0 : \forall s, \bar{s} \in [0,T], |s - \bar{s}| \leq \delta \implies ||e_h(s) - e_h(\bar{s})|| \leq \epsilon.$$

**Theorem 6** (Semidiscrete error estimate). Let $p > (1 + d)/2$. Let $h_1 > 0$ be such that $C_T h_1^{p+1/2} = \frac{1}{2} h_1^{1+d/2}$, where $C_T$ is the constant from Lemma 4. Then for all $h \in (0, h_1]$ we have the estimate

$$\max_{\vartheta \in [0,T]} ||e_h(\vartheta)||^2 \leq C_T^2 h^{2p+1}. \quad (16)$$

**Proof.** Since $p > (1 + d)/2$, $h_1$ is uniquely determined and $C_T h^{p+1/2} \leq \frac{1}{2} h^{1+d/2}$ for all $h \in (0, h_1]$. We define the propositional function $\varphi$ by

$$\varphi(t) \equiv \left\{ \max_{\vartheta \in [0,t]} ||e_h(\vartheta)||^2 \leq C_T^2 h^{2p+1} \right\}.$$

We shall use Lemma 5 to show that $\varphi$ holds on $[0,T]$, hence $\varphi(T)$ holds, which is equivalent to (16).
(i) $\varphi(0)$ holds, since this is the error of the initial condition.

(ii) **Induction step:** We fix an arbitrary $h \in (0, h_1]$. By Remark 1, there exists $\delta_0 > 0$, such that if $t \in [0, T), \delta \in [0, \delta_0]$, then $\|e_h(t + \delta) - e_h(t)\| \leq \frac{1}{2} h^{1+d/2}$. Now let $t \in [0, T)$ and assume $\varphi(t)$ holds. Then $\varphi(t)$ implies $\|e_h(t)\| \leq C_T h^{p+1/2} \leq \frac{1}{2} h^{1+d/2}$. Let $\delta \in [0, \delta_0]$, then by uniform continuity

$$\|e_h(t + \delta)\| \leq \|e_h(t)\| + \|e_h(t + \delta) - e_h(t)\| \leq \frac{1}{2} h^{1+d/2} + \frac{1}{2} h^{1+d/2} = h^{1+d/2}.$$  

This and $\varphi(t)$ implies that $\|e_h(s)\| \leq h^{1+d/2}$ for $s \in [0, t] \cup [t, t + \delta] = [0, t + \delta]$. By Lemma 4, $\varphi$ holds on $[0, t + \delta]$. As a special case, we obtain the “induction step” $\varphi(t) \implies \varphi(t + \delta)$ for all $\delta \in [0, \delta_0]$. □

5. Conclusion

We have presented the basic ideas behind the apriori analysis of nonlinear convective problems. To keep things as simple as possible, we have presented the analysis only for a space-semidiscrrete scheme, with Dirichlet boundary conditions only. The extension to mixed boundary conditions, the extension to implicit schemes via continuation, derivation of improved estimates under the assumption $f \in (C^3_b(\mathbb{R}))^d$ and the generalization to locally Lipschitz $f \in (C^2(\mathbb{R}))^d$ can be found in [4].

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References


